E Probability Current

1. Consider the time-dependent Schrödinger Equation and its complex conjugate:

$$\imath\hbar\frac{\partial}{\partial t}\psi(x,t) = H\psi(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi(x,t) \qquad \text{(E.16)}$$

$$-\imath\hbar\frac{\partial}{\partial t}\psi^*(x,t) = H\psi^*(x,t) = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V\right]\psi^*(x,t) \qquad \text{(E.17)}$$

2. Multiply Eq. (E.16) by $\psi^*(x,t)$, and Eq. (E.17) by $\psi(x,t)$:

$$\begin{aligned} \psi^*(x,t)\imath\hbar\frac{\partial}{\partial t}\psi(x,t) &= \psi^*(x,t)H\psi(x,t) \\ &= \psi^*(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\right]\psi(x,t) \end{aligned} \tag{E.18}$$

$$\begin{aligned} -\psi(x,t)\imath\hbar\frac{\partial}{\partial t}\psi^*(x,t) &= \psi(x,t)H\psi^*(x,t) \\ &= \psi(x,t)\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+V\right]\psi^*(x,t) \end{aligned} \tag{E.19}$$

3. Subtract the two equations to obtain (see that the terms involving V cancels out)

$$\begin{aligned} \imath\hbar\frac{\partial}{\partial t}\left[\psi(x,t)\psi^*(x,t)\right] &= -\frac{\hbar^2}{2m}\left[\psi^*(x,t)\frac{\partial^2}{\partial x^2}\psi(x,t)\right. \\ &- \left.\psi(x,t)\frac{\partial^2}{\partial x^2}\psi^*(x,t)\right] \end{aligned} \tag{E.20}$$

or

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\hbar}{2m\imath}\frac{\partial}{\partial x}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)-\psi(x,t)\frac{\partial}{\partial x}\psi^*(x,t)\right] \tag{E.21}$$

4. If we make the variable substitution:

$$\begin{aligned} \mathcal{J} &= \frac{\hbar}{2m\imath}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t) - \psi(x,t)\frac{\partial}{\partial x}\psi^*(x,t)\right] \\ &= \frac{\hbar}{m}\mathcal{I}\left[\psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)\right] \end{aligned} \tag{E.22}$$

   (where $\mathcal{I}[\cdots]$ represents the imaginary part of the quantity in brackets) we obtain

$$\frac{\partial}{\partial t}\rho(x,t) = -\frac{\partial}{\partial x}\mathcal{J} \tag{E.23}$$

   or in three-dimensions

$$\frac{\partial}{\partial t}\rho(x,t) + \nabla\cdot\mathcal{J} = 0 \tag{E.24}$$

5. Equation (E.24) looks like the continuity equation of a classical fluid. Hence $\mathcal{J}$ is a flux. In fact $\mathcal{J}$ is the flux associated with the probability density and is hence called the "probability current".

6. (Continuity equation of a classical fluid is basically the following: If you have a small volume element in a fluid. The change in density in that volume element is given by the amount of fluid coming into the volume element minus the amount going out. That is, flux in minus flux out. The mathematical form of this is Eq. (E.24). Hence the time-dependent Schrödinger Equation is a continuity equation.)

7. **Homework**:

   (a) Prove that $\{\nabla \cdot \mathcal{J} = 0\}$ for a *stationary state*.

   (b) Calculate the probability current for:

$$\psi = \frac{\exp\{\imath kx\}}{x} \tag{E.25}$$

   (c) Find a relation between the probability current $\mathcal{J}$ and the action function $\mathcal{S}$ that we introduced in the previous extra credit homework. Simplify this expression using Eq. (F.1). Explain your result.

   (d) For the case of the 1D barrier that we studied earlier, using the scattering perspective, calculate the probability current for ψ_1 for $x < 0$, and for $x > 0$.

