HOMEWORK SPIN OPERATORS:

1) WRITE DOWN THE MATRIX REPRESENTATION OF
   (a) $\hat{S}_z$
   (b) $\hat{S}_x$
   (c) $\hat{S}_y$

   IN THE $|S_z^+\rangle$ AND $|S_z^-\rangle$ BASIS.

   [HINT: USE THE RESOLUTION OF THE IDENTITY FOR THE EIGENSTATES OF EACH OPERATOR.]

2) AFTER YOU OBTAIN THE MATRICES ABOVE, CONSTRUCT THE COMMUTATORS; $[S_x, S_y]$, $[S_y, S_z]$ & $[S_z, S_x]$. SIMPLIFY. AND COMMENT ON YOUR RESULT.

   [NOTE: $[A,B] = AB - BA$].

D A little bit of help on the spin operators homework:

1. Consider the *ket* vectors $|+\rangle$ and $|-\rangle$. Let these *ket* vectors represent the up-spin and down-spin states of an electron along the z-orientation. (*i.e.*, $|S_z^+\rangle$ and $|S_z^-\rangle$) A state with spin = +1/2 and is represented by the vector $|+\rangle$. What is meant by this statement is that $S_z\,|+\rangle = +\hbar(1/2)\,|+\rangle$. The state with spin = -1/2 is represented by the vector $|-\rangle$. Again, this statement implies $S_z\,|-\rangle = -\hbar(1/2)\,|-\rangle$) That is, these are *eigenstates* of the S_z operator. These two vectors form a 2-dimensional vector space that is *complete* and *orthonormal*. In matrix notation, these *ket* vectors may be written as

$$|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.9}$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{D.10}$$

This is based on the isomorphism between $|+\rangle$ and x-polarized light and $|-\rangle$ and y-polarized light.

Since these two vectors form a 2-dimensional vector space that is *complete* and *orthonormal* the resolution of the identity in this space can be written as:

$$\{|+\rangle\,\langle+|\} + \{|-\rangle\,\langle-|\} = I \tag{D.11}$$

2. Using these *ket* vectors the S_z operator can be represented as follows:

$$
\begin{aligned}
S_z &\equiv S_z \left[\{ |+\rangle \langle +| \} + \{ |-\rangle \langle -| \} \right] \\
&= \left[\frac{\hbar}{2} |+\rangle \langle +| - \frac{\hbar}{2} |-\rangle \langle -| \right] \\
&= \frac{\hbar}{2} \left[|+\rangle \langle +| - |-\rangle \langle -| \right]
\end{aligned}
\tag{D.12}
$$

   (Note that the quantity in square brackets [...] on the left side in Eq. (D.12) is just the identity as per Eq. (D.11). Also note that we have used $S_z |+\rangle = +\hbar(1/2) |+\rangle$ and $S_z |-\rangle = -\hbar(1/2) |-\rangle$ to obtain Eq. (D.12). Obtain similar expressions for S_x and S_y.

3. S_z can then be written in matrix form using the *basis-ket* vectors $|+\rangle$ and $|-\rangle$ as:

$$
\begin{aligned}
S_z &\equiv \begin{pmatrix} \langle +| S_z |+\rangle & \langle +| S_z |-\rangle \\ \langle -| S_z |+\rangle & \langle -| S_z |-\rangle \end{pmatrix} \\
&= \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}
\tag{D.13}
$$

   Write down similar matrix forms for the expressions you obtained for S_x and S_y in the previous problems. These three matrices are called the Pauli-spin matrices.

4. Using the three matrices you have for S_x, S_y, and S_z, confirm that these matrices do not commute.

5. Pauli-spin matrices are 2×2 matrices. Which means they will act on 2×1 vectors. As noted earlier

$$
|+\rangle \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{D.14}
$$

and

$$|-\rangle \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{D.15}$$

And the Pauli-spin matrices can act on either these vectors or linear combinations of these vectors. Such vectors obtained from arbitrary linear combinations of $|+\rangle$ and $|-\rangle$ are called "spinors" (which comes from **spin**-vect**ors**. And in general the coefficients in front of each vector $|+\rangle$ and $|-\rangle$ can be complex.

