

12 Theory of Operators: I

1. We may recall what operators are from our earlier discussion:

$$\mathcal{O}|\eta\rangle = |\chi\rangle \quad (12.1)$$

2. Every observable quantity has an operator associated with it

3. Eigenvalues and Eigenvectors:

$$\mathcal{O}|\eta\rangle = \lambda|\eta\rangle \quad (12.2)$$

There are special vectors of this kind associated with every operator.

4. **Linear Operators:** Consider two operators \hat{A} and \hat{B} . These operators are considered “linear operators” if:

(a)

$$[\hat{A} + \hat{B}]|\eta\rangle = \hat{A}|\eta\rangle + \hat{B}|\eta\rangle \quad (12.3)$$

(b) If c is a number (may be complex) then

$$\hat{A}[c|\eta\rangle] = c\{\hat{A}[|\eta\rangle]\} \quad (12.4)$$

(c)

$$\hat{A}[|\eta\rangle + |\chi\rangle] = \hat{A}|\eta\rangle + \hat{A}|\chi\rangle \quad (12.5)$$

5. Note: The Hamiltonian operator is a linear operator. Why? Because the derivative operator is a linear operator.

6. The product of two operators \hat{A} and \hat{B} is defined as follows:

$$\hat{A}\hat{B}|\eta\rangle = \hat{A}[\hat{B}|\eta\rangle] = \hat{A}[|\chi\rangle] = |\alpha\rangle \quad (12.6)$$

where we have assumed $\hat{B}|\eta\rangle \equiv |\chi\rangle$. Note this also defines the square of an operator:

$$\hat{A}^2|\eta\rangle = \hat{A}[\hat{A}|\eta\rangle] \quad (12.7)$$

7. **Commutators:** The commutator of two operators \hat{A} and \hat{B} is defined as

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A} \quad (12.8)$$

8. Note that an operator \hat{A} *commutes* with itself since

$$[\hat{A}, \hat{A}] = \hat{A}\hat{A} - \hat{A}\hat{A} = 0 \quad (12.9)$$

9. **Anti-Commutators:** The anti-commutator of two operators \hat{A} and \hat{B} is defined as

$$[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A} \quad (12.10)$$

Note the “+”

10. **Homework:** Work out the following commutators:

(a) $\left[\frac{d}{dx}, \frac{d}{dx}\right] = ?$

(b) $\left[x, \frac{d}{dx}\right] = ?$

(c) $\left[f(x), \frac{d}{dx}\right] = ?$

(d) $\left[f(x), \frac{d}{dx}\right]_+ = ?$

(e) $\left[\frac{d}{dx}, \frac{d}{dx}\right]_+ = ?$

(f) $\left[\frac{d}{dx} + f(x)\right]^2 = ?$

11. **Eigenvalues and Eigenvectors**

$$\hat{A}|\eta\rangle = a|\eta\rangle \quad (12.11)$$

where a is a number, called the eigenvalue. $|\eta\rangle$ is the eigenvector.

12. Representing operators

- (a) Earlier we spoke about how we could “represent” vectors. That is, any vector can be represented as a linear combination of a complete set of vectors. (If this statement is not clear, please revise the linear algebra notes.)
- (b) Operators can be represented in a similar for. In fact if you have a complete set of vectors $\{|i\rangle\}$, we can write an operator as a matrix. What we mean by this is we could represent an operator using a collection of matrix elements that have the following form:

$$A_{j,l} \equiv \langle j | \hat{A} | l \rangle \quad (12.12)$$

$A_{j,l}$ is the (j, l) -th element of the matrix that is used to represent the operator \hat{A} . (Make sure to compare this with the Pauli spin matrix homework so you understand whats going on clearly.)

- (c) Does this definition make sense? $\hat{A} | l \rangle$ is another vector. You could call it $|m\rangle$ if you like. In that case the right hand side of Eq. (12.12) is the “dot” product of two vectors: $\langle j |$ and $|m\rangle$. The “dot” product of two vectors is a number. Hence the definition in Eq. (12.12) makes sense. (If these arguments are not a 100% clear to you, you need to go back and revise handout on linear algebra and also the one dealing with representation theory.)
- (d) For Eq. (12.12) to be useful we should know what \hat{A} does to $|l\rangle$ when it acts on it.