A Introductory Linear Algebra-I

1. To learn quantum mechanics, we could choose one of two options:
    (a) The historical option: we follow experimental findings between 1905 to 1922, or
    (b) concentrate on just one of the historical experiments to understand the need for a "new physics"
2. In this course we do the latter. During the first week of classes we study the Stern-Gerlach experiments, which in my opinion, fully illustrate the need for a quantum mechanical theory.
3. However, in the process of studying the results from the Stern-Gerlach experiments, we will note that these can be understood by constructing an analogy to polarized light. Furthermore, we will also see that polarized light can be treated using vector algebra. Hence we find it necessary to use vectors to understand quantum mechanics. As a result, this section serves as a review of chosen concepts from linear vector spaces and linear algebra. *We are preparing for the task that lies ahead.*
4. **Beware, there are homework problems in here. Hence I would suggest that you do not skip any items!!**
5. The discussion will start from the very basics.

6. We will start with simple concepts from three-dimensional vector spaces.

7. In 3 D a vector has three components:

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \tag{A.1}$$

$\hat{i}$, $\hat{j}$ and $\hat{k}$ are unit vectors in the x, y and z directions respectively. $\hat{i}$, $\hat{j}$ and $\hat{k}$ may also be called basis vectors or just bases and this is a terminology that we will use often. Vectors can be represented in the following form:

$$\vec{r} \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{A.2}$$

8. The vector is fully described by: (i) its magnitude, and (ii) its direction. Compare this with a scalar number that only has a magnitude.

9. The magnitude of the vector $\vec{r}$ can be calculated by performing the following mathematical operation:

$$|\vec{r}| \equiv \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{|x|^2 + |y|^2 + |z|^2} \tag{A.3}$$

where we have introduced the definition of the “dot” product of two vectors:

$$\begin{aligned} \vec{a} &= a_1\hat{i} + a_2\hat{j} + a_3\hat{k} \\ \vec{b} &= b_1\hat{i} + b_2\hat{j} + b_3\hat{k} \\ \vec{a}\cdot\vec{b} &= a_1b_1 + a_2b_2 + a_3b_3 \end{aligned} \tag{A.4}$$

10. The “dot” product is a very fundamental concept, and very useful for our further development hence lets look at it a little closely.

11. Note that by the definition of matrix multiplication, I can write the dot product of $\vec{a}$ and $\vec{b}$ as:

$$\vec{a} \cdot \vec{b} = (a_1 \; a_2 \; a_3) \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{A.5}$$

Homework: Evaluate Eq. (A.5) using the laws of matrix multiplication and make sure you get the same result as in Eq. (A.4). To complete this homework we may need to review matrix multiplication. Please do so, or talk to the instructor to make sure that this is not a problem.

12. **Homework: Multiply the two matrices:**

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} and B = \begin{pmatrix} 1 & 4 & 2 & 5 \\ 3 & 6 & 9 & 10 \end{pmatrix} \tag{A.6}$$

Pay careful attention to the order in which these matrices are multiplied. What A multiplied by B? What is B multiplied by A? Based on this exercise what can you say about the order in which matrices are multiplied?

A.1 The Dual Space

13. The object $(a_1\ a_2\ a_3)$ in Eq. (A.5) can be referred to as the *dual-space analogue* of the object:

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{A.7}$$

14. Note that every vector has its *dual-space analogue*, and for real numbers, the dual space analogue is just the transpose of the vector.

15. So we may write:

$$\vec{a} \cdot \vec{b} = \vec{a}^T \vec{b} \tag{A.8}$$

16. However, in general the components of the vector $\vec{a}$ need not be real (they could be complex!!) and then the dual space analogue is not simply the transpose. In the complex case the dual space analogue of the vector $\vec{a}$ is defined as:

$$\vec{a}^\dagger = (a_1^*\ a_2^*\ a_3^*) \tag{A.9}$$

where the “superscripted *” imply *complex conjugation*. (Do you know what that means? If not ask the instructor and he will explain it to you.)

17. **Homework: Show that the definition in Eq. (A.9) is consistent with the dot product definition in Eq. (A.4). Hint: Evaluate the quantity $\vec{a}^\dagger\vec{a}$ using multiplication of matrices as you did in Eq. (A.5). Show that you get the same result as Eq. (A.4).**

18. The dual space analogue is a very important concept in quantum theory, as we will see later. For now we understand the dual space analogue is the complex conjugated transpose of a vector.

19. The basis vectors, $\hat{i}$, $\hat{j}$ and $\hat{k}$ that we introduced earlier, have a very important property and that is the "dot" product of each of these with itself is equal to 1, but the the dot product of one with any of the others is always 0 (zero). This means:

    $$\hat{i} \cdot \hat{j} = \hat{i} \cdot \hat{k} = \hat{j} \cdot \hat{k} = 0 \qquad \text{(A.10)}$$

    but

    $$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \qquad \text{(A.11)}$$

    These relations make the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ an *ortho-normal* basis set. The term *ortho-normal* has two parts: the first part *ortho* is part of the word *orthogonal*. Two vectors are said to be *orthogonal* when they are at 90 degrees. (Note that the vectors $\hat{i}$, $\hat{j}$ and $\hat{k}$ are at 90 degrees to each other, as is required from the fact that these are unit vectors along the x, y and z directions.) The second part of *ortho-normal* is the word *normal*: a vector is normalized when its magnitude is 1.

B Introductory Linear Algebra-II: *Dyads*

1. We shall now introduce a very important mathematical quantity called a *dyad*. This is fundamental in the theory of linear vector spaces, and turns out to be of great importance in quantum mechanics. (Make sure you got the exercise on matrix multplication in the previous section before you proceed into this one.)

    - Before we introduce the dyad, lets consider the dot product in Eq. (A.5). The product of a (1×3) matrix with a (3×1) matrix leads to a (1×1) (*i.e.* a scalar number) matrix.
    - What would happen if we had things backwards?

      $$\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} (a_1 \; a_2 \; a_3) \tag{B.1}$$

      This represents a (3×1) matrix multiplied with a (1×3) matrix and hence the result should be a (3×3) matrix !!
    - The quantity in Eq. (B.1) is called the *dyadic product* or *outer product* of the two vectors. By contrast the dot product in Eq. (A.5) is also known as the *inner product* of two vectors.

2. To understand some of the implications of the dyadic product (there will be other implications that we will realize when we introduce the Dirac notation) let us consider the basis vectors in 3-dimensional space:

$$\vec{i} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; ; \; \vec{j} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \; ; \; \vec{k} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{B.2}$$

3. The dyadic product:

$$\vec{i}\vec{i}^\dagger = \begin{pmatrix} 1\;0\;0 \\ 0\;0\;0 \\ 0\;0\;0 \end{pmatrix} \tag{B.3}$$

   that is a matrix. Similarly the dyadic products $\vec{i}\vec{i}^\dagger$, $\vec{i}\vec{j}^\dagger$, $\vec{i}\vec{k}^\dagger$, $\vec{j}\vec{i}^\dagger$, $\vec{j}\vec{j}^\dagger$, $\vec{j}\vec{k}^\dagger$, $\vec{k}\vec{i}^\dagger$, $\vec{k}\vec{j}^\dagger$, $\vec{k}\vec{k}^\dagger$ are all matrices with 1 at one of the positions in the matrix and zeroes everywhere else.

4. **Homework:**

   (a) **Write down all the dyadic products** $\vec{i}\vec{i}^\dagger$**,** $\vec{i}\vec{j}^\dagger$**,** $\vec{i}\vec{k}^\dagger$**,** $\vec{j}\vec{i}^\dagger$**,** $\vec{j}\vec{j}^\dagger$**,** $\vec{j}\vec{k}^\dagger$**,** $\vec{k}\vec{i}^\dagger$ $\vec{k}\vec{j}^\dagger$**,** $\vec{k}\vec{k}^\dagger$

   (b) **Homework: Show that:**

$$\vec{i}\vec{i}^\dagger + \vec{j}\vec{j}^\dagger + \vec{k}\vec{k}^\dagger = \begin{pmatrix} 1\;0\;0 \\ 0\;1\;0 \\ 0\;0\;1 \end{pmatrix} \tag{B.4}$$

   The matrix on the right hand side in this equation is called the identity or unit matrix. This relation is called the *resolution of the identity* and we will have a great deal of need for this expression.

(c) Using the results obtained above, show that the matrix:

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \tag{B.5}$$

can we written as a linear combination $\vec{i}\vec{i}^\dagger$, $\vec{i}\vec{j}^\dagger$, $\vec{i}\vec{k}^\dagger$, $\vec{j}\vec{i}^\dagger$, $\vec{j}\vec{j}^\dagger$, $\vec{j}\vec{k}^\dagger$, $\vec{k}\vec{i}^\dagger$ $\vec{k}\vec{j}^\dagger$, $\vec{k}\vec{k}^\dagger$.

5. Note: You have just performed a very important exercise. An important essence of quantum mechanics lies in this exercise.

6. Firstly, the resolution of the identity is also known as the completeness relation. Any vector can be written as a linear combination of an *ortho-normal* and *complete* set of vectors. As the set $\hat{i}$, $\hat{j}$ and $\hat{k}$ are *ortho-normal* (as seen in point 19 of Section A) and *complete* (as seen in point 4b) any vector in three-dimension can be written as a linear combination of these vectors.

7. Secondly, we also note from point 4 that any matrix is writable as a linear combination of the *dyadic*-basis: $\vec{i}\vec{i}^\dagger$, $\vec{i}\vec{j}^\dagger$, $\vec{i}\vec{k}^\dagger$, $\vec{j}\vec{i}^\dagger$, $\vec{j}\vec{j}^\dagger$, $\vec{j}\vec{k}^\dagger$, $\vec{k}\vec{i}^\dagger$ $\vec{k}\vec{j}^\dagger$, $\vec{k}\vec{k}^\dagger$.