

G A brief note on change of basis

1. We spoke about the definition of a wavefunction. It is the projection of the abstract “ket” $|\psi\rangle$ onto the coordinate representation “ket”, *i.e.*, $\langle x|\psi\rangle \equiv \psi(x)$, the wavefunction.
2. But then the abstract “ket” $|\psi\rangle$ can be represented using any *complete set of kets*, and we have chosen above to use the coordinate representation, which we know to be complete from the identity:

$$\int dx |x\rangle \langle x| = 1 \quad (\text{G.1})$$

But we could have chosen to use some other set, for example the momentum representation, since:

$$\begin{aligned} |\psi\rangle &\equiv \int dx |x\rangle \langle x|\psi\rangle = \int dx |x\rangle \psi(x) \\ &\equiv \int dk |k\rangle \langle k|\psi\rangle = \int dk |k\rangle \tilde{\psi}(k) \\ &\equiv \sum_i |i\rangle \langle i|\psi\rangle = \sum_i |i\rangle c_i. \end{aligned} \quad (\text{G.2})$$

where $\psi(x)$, $\tilde{\psi}(k)$ and c_i are just projects of the abstract “ket” $|\psi\rangle$ onto the “axes” of the chosen representation.

3. The statement along with our discussion on measurement as a “dot” product may be viewed to imply that, in fact, it is the projection of $|\psi\rangle$ in some basis that is measured (or calculated) and from such a projection we (sometimes) struggle to recreate the entire “ket” $|\psi\rangle$.
4. That begs the question, how do these projections relate to each other. That is how is $\psi(x)$ related to $\tilde{\psi}(k)$ and c_i and vice versa.

5. This is the question we address here as part of discussion on “change of basis”.
6. Consider the following sequence of arguments:

$$\begin{aligned}\psi(x) &= \langle x | \psi \rangle \\ &= \langle x | I | \psi \rangle \\ &= \langle x | \int dk |k\rangle \langle k | | \psi \rangle\end{aligned}\tag{G.3}$$

where we have inserted the identity in terms of the momentum representation. Now, since $\langle x |$ is a vector and $\int dk$ is essentially a continuous summation, we can take the vector inside the summation (to be discussed in class), leading to:

$$\psi(x) = \int dk \langle x | k \rangle \langle k | | \psi \rangle\tag{G.4}$$

Now, $\langle x | k \rangle$ is the projection (or representation) of the momentum eigenstate in the coordinate representation. That is, $\langle x | k \rangle$ is a function of x , that is also an eigenstate of the momentum operator.

7. So, what is $\langle x | k \rangle$? Well, $\langle x | k \rangle = \exp(ikx)$, because $\exp(ikx)$ is an eigenstate of the momentum operator and is a function of x . Read the previous point again to make sure you understand this.
8. That leads to:

$$\begin{aligned}\psi(x) &= \int dk \exp(ikx) \langle k | | \psi \rangle \\ &= \int dk \exp(ikx) \tilde{\psi}(k)\end{aligned}\tag{G.5}$$

9. Some of you might recognize that the above equation is a Fourier transform. Indeed, one needs to construct a Fourier transform to

convert a wavefunction into its momentum space analogue and vice versa. But this is now an example of a change of basis. That is if we have $\tilde{\psi}(k)$, we can always use the above equation to get $\psi(x)$, that is the projection of $|\psi\rangle$ on to a different basis.

- 10. Homework: Work out the transformation of $\psi(x)$ onto some discrete basis $\{|i\rangle\}$.**