

Section 31 Revisited: Frege's Elucidations

Joan Weiner

University of Wisconsin, Milwaukee

Section 31 of Frege's *Basic Laws* is titled "our simple names denote something (*bedeuten etwas*)".¹ Any sophisticated contemporary reader knows, in advance of reading section 31, that there must be something wrong with at least one part of the argument. For one of the simple Begriffsschrift names is supposed to name a second-level function whose value, given a first-level function as argument, is the course-of-values of that first level function.² And there can be no such function. The assumption that each function has a course-of-values is the source of the inconsistency of the logic of *Basic Laws*. The reader who knows this but has not yet read section 31 will expect to find an apparently good argument that the Begriffsschrift course-of-values function name designates a second-level function; that is, an argument that *would be* perfectly good were Frege's assumption correct. But such a reader will be surprised. Instead of an apparently good argument, the reader will find an apparently circular argument that, circularity aside, depends on assumptions that seem inconsistent with Frege's explicit statements about the nature of functions.³ The puzzle that immediately confronts the reader is not so much a puzzle about where Frege went wrong as it is a puzzle about how Frege can have believed that his proof works.

Attempts to solve this puzzle have generated a substantial secondary literature.⁴ Although there are disagreements in this literature about what mistakes or confusions led Frege to formulate his peculiar argument, there is also a large area of consensus. There is consensus that section 31 is meant to carry out the basis case of an inductive metatheoretic proof that all Begriffsschrift expressions have Bedeutung.⁵ And there is

My thanks to Mark Kaplan for helpful discussions and criticisms of earlier drafts and to Thomas Ricketts for correspondence about some of the topics discussed in this paper.

¹ Citations of Frege's published work, with the exception of *Begriffsschrift*, are all to page numbers or section numbers of the original publication. For *Begriffsschrift* and the unpublished work both English and German citations are included.

² "Course-of-values" is Montgomery Furth's translation of "*Wertverlauf*". Since I have chosen to use Furth's translation in my quotations from *Basic Laws*, I will, except where explicitly noted, be using his translations in my own prose as well.

³ Although circular arguments may not invariably be problematic, there is consensus that the circularity in this case is problematic.

⁴ See, for example, (Parsons, 1965), (Resnik, 1986), (Thiel, 1996), (Heck, 1997).

⁵ In this paper I will be discussing not only the early sections of *Basic Laws* but also some of Frege's papers, including "On Sense and Meaning" and some of the secondary literature. Where Montgomery Furth's translation uses 'denotation' and its cognates, the translators of "On Sense and Meaning" use 'meaning' and its cognates and Michael Resnik, whose interpretation I will be discussing uses 'reference'. Moreover, the choice of translation in many cases carries with it a view about how Frege's writings should

consensus that there are multiple confusions and mistakes in Frege's argument. I believe that, in both cases, the consensus view is mistaken. It is the first view—that section 31 is meant to carry out the basis case of an inductive metatheoretic proof—that forces most philosophers who have written about section 31 to attribute confusions to Frege. I will argue that Frege's writings provide ample straightforward evidence that this view is incorrect: the argument in section 31 is meant neither as a metatheoretic proof nor as the basis case of an inductive argument. In this paper I will offer an alternative reading that takes seriously, as the standard reading does not, Frege's explicit remarks about the nature of his project and his logic. I will end with an explanation of how this reading solves at least one of the exegetical puzzles in section 31: on my reading, there is no circularity in the arguments of section 31.

The issue, as I have described it so far, may seem to be one of significance only to those interested in the exegesis of a small isolated piece of text, a piece of text familiar only to specialists. In fact, however, what is also at stake is how to understand Frege's conception of logic. On the reading that until recently has been standard, one of Frege's contributions to the contemporary understanding of logic was the introduction, not just of a formal language, but of a metatheory. The most unequivocal support for this reading is supposed to be found in the early sections of *Basic Laws*, which include section 31, along with "On Sense and Meaning". I will argue that, far from providing unequivocal support, these texts actually undermine the metatheory-reading. When we look at these traditional sources of the metatheory-reading with an eye toward explaining section 31, we will see that they provide a powerful argument for rejecting the metatheory-reading.

One of the salutary consequences of this rejection, as I will show, is that it allows us to make sense of some of the initially puzzling features of section 31. Moreover, when we attempt to make sense of section 31, we will be steered toward passages that provide an outline of Frege's conception of logic and its relation to language. Although this conception differs from those popular today it is, in my view, neither archaic nor inferior to contemporary conceptions. As I have argued elsewhere, some of the problems that

be interpreted and one of my aims is to address the issue of which view is correct. Because of this, I have chosen not to alter any of the translations of these terms in the passages that I quote. In order to minimize confusion, however, I have included the German expressions as well. In my own prose, I have sometimes used the translation used by a writer I am discussing and other times I have chosen not to adopt any of the translations but to use the expression 'has Bedeutung' as if it were an English expression. I apologize for any confusion or infelicity that results.

bother philosophers today are best addressed by taking into account some of Frege's neglected insights about language and logic.⁶

I. Section 31 and the Metatheory-Reading

Although there are a number of statements in §31 that are likely to puzzle any reader, §31 presents special difficulties for advocates of the metatheory-reading. One difficulty is that Frege's statement of what is required for the proof in §31 conflicts with what, according to the metatheory-reading, should be required of such a proof. To see this, let us consider some central features of the metatheory-reading. Michael Dummett writes:

For Frege, the reference of an expression is an extra-linguistic entity, and, in the informal semantics, or model theory, which has been developed from his ideas, an interpretation associates with each individual constant, predicate, etc., of the language a non-linguistic entity of a suitable type. ((Dummett, 1978), p. 123)

On this reading, Frege means to be setting out an intended interpretation for the formal logic in the early sections of *Basic Laws*. To hold that Frege is introducing a metatheory in these sections we need not go so far as to say that the intended interpretation actually is a model theoretic interpretation of his formal language. Even Dummett writes only that model theory was "developed from his ideas". But to hold this view we do need to regard Frege's introduction of his logical language as something more than an explanation of what its symbols mean, of how to understand the new language. As Dummett indicates, we need to regard the introductions of the symbols as associating them with extra-linguistic entities.

Of course, the mere association of symbols with extra-linguistic entities need not constitute part of a metatheory. One does not enter the realm of metatheory by naming the family pet. What makes such statements a part of a metatheory is their use in proofs. On the metatheory-reading the proofs of *Basic Laws* are not limited to those expressed in Begriffsschrift. They include, also, proofs about the logical language and its interpretation. These are proofs in which statements about his symbols' referring to extra-linguistic entities play a role; proofs involving an ineliminable use of a truth predicate for the logical language. Such proofs are stated in natural language. On this reading, the task of §31 is to set out part of one of these proofs.

The proof in question, according to the metatheory-reading, is an inductive proof, found in §§28-31, that the intended interpretation associates appropriate entities with

⁶ I have suggested this in "Understanding Frege's Project", forthcoming in *The Cambridge Companion to Frege*. More recently, I have tried to show how issues concerning vagueness are best understood by incorporating Frege's insights into the relation of logic and language in "Science and Semantics: the Case of Vagueness" (in preparation).

each expression of Begriffsschrift. In the earlier sections Frege proves the inductive step: that any complex term constructed from terms that are associated with appropriate entities will, itself, designate an appropriate entity. And the task of §31 is to prove the basis case: that each of his primitive terms is associated with an appropriate entity. Supposing the metatheory-reading is correct, how would we expect Frege to carry out the task of §31?

It is important to begin by noting that all primitive Begriffsschrift expressions are function-signs. Begriffsschrift has no simple proper names. Thus there are no stipulations that particular proper names name particular objects.⁷ Since the primitive Begriffsschrift expressions are function-names, Frege should assign, by stipulation, a function to each primitive function sign. The burden of §31 is, it seems, to show that these stipulations are successful. But one would not expect this to be a difficult task and, in particular, one would not expect this task to require a proof. All that seems to be required is an explanation, in everyday language, of which function is assigned to each primitive function-sign. If, for example, one were to decide to introduce a primitive term for the successor function on the natural numbers, it would seem sufficient to stipulate that the term in question designates this function.⁸ Given the requisite stipulation, no further proof should be required (although, if we are careless in our choice of axioms, the axioms may turn out to be false under this interpretation).⁹ Thus there is something peculiar about the activity of offering a proof that the intended interpretation associates the appropriate sort of entity with each primitive Begriffsschrift name. What is the point of offering such proofs in §31?

⁷ In fact, it is probably also worth noting that, in Begriffsschrift, unlike contemporary logical notations, there are no non-logical constants—although it is evident that Frege thought that non-logical constants should be added in order to bring the tools of logic to bear on other disciplines. But only once Begriffsschrift is supplemented in this way would it be possible to give the sort of interpretation that is part of contemporary logic.

⁸ I choose this example for its familiarity to contemporary readers. But I do not mean to suggest that Frege might introduce this function as an interpretation of a primitive logical sign. He would not because he does not regard the successor function as a primitive logical function. Frege's primitive logical functions are less familiar, not only to contemporary readers but also to Frege's intended audience. However, it is evident from the discussions in which he introduces his primitive function signs (see, *Basic Laws I*, sections 5-10) that, except for the second-level course-of-values function, he does not regard this unfamiliarity as a reason for requiring special proofs.

⁹ There is, in particular, no role for such proofs on the contemporary understanding of logic. An interpretation assigns each non-logical constant or function name an appropriate entity by stipulation. No proof is required. Nor is there any role for a proof that an interpretation assigns an appropriate entity to the sorts of Begriffsschrift signs that are of concern in section 31. For these are not assigned entities at all by contemporary model theoretic interpretations. Their contribution to the truth-values of sentences in which they appear is provided by a definition of truth under interpretation.

One obvious answer is that an attempt at stipulation could go wrong. After all, the function I mentioned in the above example, the successor function, is rather different from the functions that Frege assigns to his primitive symbols. Frege might have worried (and with good reason) that there are problems with defining these peculiar functions—he might have worried, that is, that his description of the functions did not suffice to specify a unique value for each argument. The real worry, of course, is about Frege's attempt to describe a particular function: the second-level course-of-values function. There may be no function that satisfies Frege's description. Thus Frege seems to need an argument that his attempt at a natural language description of this function really does pick out a function. If it does, then the Begriffsschrift second-level course-of-values function name really picks out a function. Moreover, there is some support for the view that the point of §31 is to take care of this problem. For, while Frege discusses all his primitive function-names in §31, most of his attention is devoted to the second-level course-of-values function name. On this reading, the important proof is one case of a general proof that all primitive Begriffsschrift terms have *Bedeutung*. But can we read §31 as an attempt to prove that all primitive Begriffsschrift terms have *Bedeutung* in this way—that is by proving that the everyday language definitions of the functions these terms are to designate really do pick out functions?

There are several problems with such a reading. One of these is that the worry does not seem the sort of worry that can be addressed by metatheoretic proof. The issue, in fact, is not a metatheoretic issue at all. Frege has attempted to describe functions in natural language and he has stipulated that these are named by certain Begriffsschrift expressions. But the worry is not about the stipulation. The worry is about his attempts at descriptions of functions. The issue is whether these attempts succeed. If so then, of course, all his primitive Begriffsschrift terms have *Bedeutung*. If not, they do not. It is difficult to see how metatheory has anything to contribute.

Another problem with this sort of reading is that Frege's account of what must be shown seems incorrect. He writes:

In order now to show, first, that the function-names " ——— ξ " and " ——— ξ " denote something (*etwas bedeuten*), we have only to show that those names succeed in denoting (*bedeutungsvoll sind*) that result from our putting for " ξ ", the name of a truth-value (we are not yet recognizing other objects). (BLA vol. I, §31).

There are several puzzles here. One of these is that Frege does not talk about associating entities with his symbols. This might seem reasonable were function names the only names at issue in this passage. For, one might argue, the association of a function with a function-name does not require talk about entities or functions—it requires only a

definition of the function, that is, an indication of what values it has for each argument. But Frege is not talking only about function-names in this passage. Some of the names mentioned in the above passage are object-names: the results of putting a name of a truth-value for " ξ " in " $\text{---} \xi$ " and " $\text{---} \xi$ ". On the metatheory-reading, what needs to be shown of an object name is that the interpretation associates it with an *object*. But, as his surprising choice of words indicates—there is no indication in the German expression '*bedeutungsvoll sind*' that any object or relation between a name and object is involved—Frege does not acknowledge this.¹⁰

One might suspect that this is just an odd choice of words. But this choice of words is entirely in line with Frege's explanation, in §29, of when an object name denotes something. He writes:

A proper name has a denotation (*hat eine Bedeutung*) if the proper name that results from that proper name's filling the argument-places of a denoting name (*eines bedeutungsvollen Namens*) of a first-level function of one argument always has a denotation (*eine Bedeutung hat*), and if the name of a first-level function of one argument that results from the proper name in question's filling the ξ -argument-places of a denoting name (*eines bedeutungsvollen Namens*) of a first-level function of two arguments always has a denotation (*eine Bedeutung hat*), and if the same holds also for the ζ -argument-places. (BLA vol. I, §29)

There is no explicit mention that there must be objects that the proper names denote (or functions that the function names denote). Even worse, the explanation of when a proper name has *Bedeutung* seems circular. A proper name has *Bedeutung*, provided certain other terms—terms formed by putting the proper name in the argument place of a first-level function sign that has *Bedeutung*—do. So it seems that the explanation of when a proper name has *Bedeutung* is parasitic on the explanation (which appears in the previous paragraph of §29) of when a first-level function sign has *Bedeutung*. Yet that explanation is

¹⁰Although it is possible that this use of words is an oversight, it seems unlikely. For, in his discussions of the notions of function and concept, Frege exploits the different significance of various uses of 'bedeuten' and its cognates. In some unpublished notes that are estimated by the editors of Frege's *Nachlass* to have been written sometime between 1892-1895, the following appears:

Indeed we should really outlaw the expression 'the meaning of the concept-word A' [*die Bedeutung des Begriffsworts A*] because the definite article before 'meaning' [*Bedeutung*] points to an object and belies the predicative nature of a concept. It would be better to confine ourselves to saying 'what the concept word A means' [*was das Begriffswort A bedeutet*] (PW p. 122/NS p. 133).

In a later letter to Russell, Frege writes:

we cannot properly say of a concept name that it means something [*dass er etwas bedeute*]; but we can say that it is not meaningless [*dass er nicht bedeutungslos sei*]. (PMC p. 136/BW p. 219).

Moreover, in section 31 Frege in fact *does not* argue that each proper name is associated with an object. He shows this for proper names that are truth-value names but not for proper names of courses-of-values that are not truth-values.

A name of a first-level function of one argument has a *denotation* (*denotes* something, succeeds in *denoting*) [*hat dann eine Bedeutung (bedeutet etwas, ist bedeutungsvoll)*] if the proper name that results from this function-name by its argument-places' being filled by a proper name always has a denotation if the name substituted denotes something (*etwas bedeutet*). (BLA vol. I, §29)

Here Frege indicates that a first-level function sign of one argument has *Bedeutung* provided certain proper names do.

The circular character of this explanation is especially puzzling because, if Frege means to be giving an interpretation of his language that associates names with extra-linguistic entities, it seems both unnecessarily complicated and incorrect.¹¹

Frege's treatment of the second-level course-of-values function sign differs, in exactly the same way, from what the metatheory-reading leads us to expect. On the metatheory-reading, Frege ought to argue that his second-level course-of-values function sign is, by stipulation, a *Begriffsschrift* name for the function that, given a first-level function as argument, yields its course-of-values as its value. For, although he is somewhat uncomfortable with this, Frege does assume in *Basic Laws* that each first-level function has a course-of-values. And this assumption is, from the contemporary point of view, enough to establish that the sign in question is associated with the appropriate sort of entity. Yet Frege does not appeal to this assumption in §31. Nor does he attempt to prove that his assumption is correct—either in §31 or anywhere else.

Does §31 contain a different sort of argument that there is a second-level course-of-values function? Frege argues that his second-level course-of-values function sign has *Bedeutung* because the proper names formed by completing it—that is, by filling in the argument place of this second-level function sign with a sign for a first-level function of one argument—satisfy the condition described in the above quotation.¹² The fact that Frege was uncomfortable with his assumption that each first-level function has a course-of-values does not explain this odd argument. Even if his discomfort prevented him from relying on this assumption, why would he have thought that the strategy employed in §31 could work?

The evidence of these sections in isolation is that, whatever Frege means by the interchangeable expressions 'has a denotation (*hat eine Bedeutung*)', 'denotes something' (*bedeutet etwas*) and '*ist bedeutungsvoll*', it is not '*associated by the interpretation with*

¹¹ It seems unnecessarily complicated since there is no obvious reason why Frege should not simply say that a first-level function name has a *Bedeutung* just in case it names a first-level function. It also seems, incorrect because a first-level function must be defined for all objects, not just those that have *Begriffsschrift* names. This issue will be examined in more detail shortly.

¹² Actually, this is a bit over simplified. He does not argue that this holds for *every* first-level function name that has *Bedeutung*, but only that it holds for the primitive first-level function names.

the appropriate sort of entity'. Thus §§28-31 present a number of difficulties for the metatheory-reading. This is no reason, in itself, to reject the metatheory-reading. After all, these are notoriously difficult sections. Moreover, there are other passages from the early sections of *Basic Laws* that appear to provide direct support of the metatheory-reading.

One apparent source of direct support is Frege's introduction of his generalized notion of function in sections 1 and 2. Functions, given Frege's generalized notion, include not only recognizable mathematical functions but also concepts and relations. Concepts and relations are functions that take arguments to truth-values. Truth-values are introduced in § 2, where he writes:

I say: the names " $2^2 = 4$ " and " $3 > 2$ " denote (*bedeuten*) the same truth-value, which I call for short *the True*. Likewise, for me " $3^2 = 4$ " and " $1 > 2$ " denote (*bedeuten*) the same truth-value, which I call for short *the False*, precisely as the name " 2^2 " denotes (*bedeutet*) the number four. Accordingly I call the number four the *denotation (Bedeutung)* of " 4 " and of " 2^2 ", and I call the True the denotation (*Bedeutung*) of " $3 > 2$ ".

In this passage Frege seems to be saying explicitly that, at least for expressions that are proper names (among which are sentences), having *Bedeutung* amounts to being associated with an extra-linguistic entity. In a footnote he refers his readers to an earlier essay "On Sense and Meaning", a work in which he makes similar remarks about *Bedeutung*.

The supporter of the metatheory-reading might argue that these passages show that the problem with §31 is not a problem with the metatheory-reading, it is a problem with Frege's discussions in §§28-31 of *Basic Laws*. It is true that Frege's answer to the question "when does a name denote (*bedeutet*) something?" in §29 involves no allusion to extra-linguistic entities. Yet, in sections 1 and 2 Frege does seem to indicate that to have *Bedeutung* simply is to be associated with the appropriate sort of extra-linguistic entity. The metatheory-reading, one might argue, is not a *reading* of the above passage, it is simply a statement of what Frege explicitly says.

But this argument is too quick. It is central to the metatheory-reading that statements about *Bedeutung* play a particular role in Frege's project—that they are used in actual proofs. And the explicit statements from sections 1 and 2 of *Basic Laws* do not go this far. The supporter of the metatheory-reading might respond that these sections are really the wrong place to look for a detailed explanation of the role that the notion of *Bedeutung* plays in Frege's project. As a footnote in section 2 suggests, the reader who wants a more complete explanation of this notion must turn to the work in which it is introduced, "On Sense and Meaning". Indeed, it is this paper that, on metatheory-reading, sets out the background against which the early sections of *Basic Laws* must be

understood—it is this paper that sets out the beginning of Frege's theory of reference. Moreover, some of Frege's comments in "On Sense and Meaning" have been taken to show why he felt a need for metatheoretic proof. Before we turn to the actual details of section 31, it will help to begin with an examination of these earlier writings. Let us turn, then, to the apparent need for metatheoretic proof that is supposed to come out in "On Sense and Meaning".

II. "On Sense and Meaning" and the Metatheory-Reading

Although there is no explicit statement in "On Sense and Meaning" about the need for metatheoretic proofs, the supporter of the metatheory-reading may argue that the need follows almost immediately from something that Frege does say explicitly. Frege writes:

A logically perfect language (*Begriffsschrift*) should satisfy the conditions, that every expression grammatically well constructed as a proper name out of signs already introduced shall in fact designate an object, and that no new sign shall be introduced as a proper name without being secured a meaning (*Bedeutung*). ((Frege, 1892a), p. 41)

He goes on to say:

The logic books contain warnings against logical mistakes arising from the ambiguity of expressions. I regard as no less pertinent a warning against apparent proper names having no meaning (*Bedeutung*). The history of mathematics supplies errors which have arisen in this way. ((Frege, 1892a), p. 41).

One might well expect this issue to come up in *Basic Laws*, which is, after all, a logic book. And it does. In *Basic Laws*, Frege explicitly identifies as a basic principle "that every correctly formed name denotes something (*etwas bedeuten*)" (BLA vol. I, p. xii).¹³ Moreover, if this is a basic principle, and a principle that has been violated both in logic books and in the history of mathematics, one might expect Frege to offer proof that he is not, himself, violating the principle. Michael Resnik writes:

No methodological principle was more important to Frege than the one at stake in these passages: *in a properly constructed scientific language every name (including function-names as well as object-names) must have a reference*. In his eyes the repeated failures of his fellow mathematicians to be certain of satisfying this tenet was one of their most grievous errors. Thus it was entirely in keeping with this that he proved that every name in his own system has reference. ((Resnik, 1986), p. 177).

If Resnik is right about this, then the above passages from "On Sense and Meaning", as well as the passages from the first two sections of *Basic Laws*, seem to tell us what is required of such proofs. For these passages seem unambiguously to support the reading

¹³ This principle, of course, is more general than the principle introduced in "On Sense and Meaning", for Frege uses 'name' in *Basic Laws*, to include the sorts of expressions that we refer to today as logical constants.

on which to say that a proper name has *Bedeutung* is to say something about a relationship between that name and an object.

But there are problems for Resnik's claim. We have already seen one of these. Frege simply does not do what he ought to do: he does not attempt to show that each primitive name is correlated with an extra-linguistic entity. There is also another serious problem. The views from "On Sense and Meaning" along with Frege's oft-stated comments about primitiveness, commit him to denying that there is any way to give the sort of metatheoretic proof that, according to Resnik, Frege wants to give.¹⁴

To see this, we need to begin by looking more closely at Frege's formulation of the worry that the principle is supposed to allay. The first formulation of the principle appears in a passage about a page earlier than the passage Resnik quotes. Frege writes:

Now languages have the fault of containing expressions which fail to designate an object (although their grammatical form seems to qualify them for that purpose) because the truth of some sentence is a prerequisite. ((Frege, 1892a), p. 40).

There is an important difference between this passage and the later passage that Resnik quotes. Frege's worry, as he describes it in this passage, is not simply that languages contain proper names that fail to designate objects. It is, rather, that languages contain proper names that fail to designate objects *because the truth of some sentence is a prerequisite*. The example that Frege uses to illustrate the problem is the following sentence:

Whoever discovered the elliptic form of the planetary orbits died in misery. The problem arises with the expression 'whoever discovered the elliptic form of the planetary orbits'. Whether or not this expression designates some object, Frege says, depends on whether or not the following sentence is true:

There was someone who discovered the elliptic form of the planetary orbits. ((Frege, 1892a), p. 40).

But, assuming the sentence *is* true, why is there a problem? One reason, Frege tells us, is that if this is so the denial of the original sentence is not,

Whoever discovered the elliptic form of the planetary orbits did not die in misery

but rather,

¹⁴ I say that Frege is committed to denying this, not that he actually did deny it. It is no surprise that he did not actually deny it. After all, it is not as if there were a pre-"On Sense and Meaning" articulation of this view available for Frege to attack. It is only our post-Tarski sensibility that makes this seem an obvious issue.

Either whoever discovered the elliptic form of the planetary orbits did not die in misery or there was nobody who discovered the elliptic form of the planetary orbits.

In other words, the natural language sentence contains a tacit presupposition: that there was someone who discovered the elliptic form of the planetary orbits. This, Frege says, arises from an imperfection of language—an imperfection that is to be avoided in a logically perfect language.¹⁵

But why is this to be avoided in a logically perfect language? The answer lies in Frege's view of the purpose of logically perfect language. When he first introduces his logical notation in *Begriffsschrift*, Frege characterizes it as a tool invented for "certain scientific purposes" and, he adds, "one must not condemn it because it is not suited to others" (BEG, p. 6/BS xi). The purposes for which it is not suited are the everyday uses that we make of natural language. What are the scientific purposes for which *Begriffsschrift* is suited?

Frege's *Begriffsschrift* is introduced as a tool for expressing proofs that can play a particular sort of role: proofs that give us "a basis upon which to judge the epistemological nature of the law that is proved" (BLA vol. I, p. vii). The particular laws with which Frege is concerned are, of course, the laws of arithmetic. He wants to show that the truths of arithmetic are analytic. The enterprise is, as he characterizes it in *Begriffsschrift*, to show that one can prove the truths of arithmetic "with the sole support of those laws of thought that transcend all particulars" (BEG, p.5/BS x). It is, as he says in the preface to *Basic Laws*, to show "that arithmetic is a branch of logic and need not borrow any ground of proof whatever from either experience or intuition" (BLA vol. I, §0). One problem with using natural language proofs to show this is that tacit presuppositions on which these proofs depend may require some ground either from experience or intuition. Frege says that it is

precisely the presuppositions made tacitly and without clear awareness that obstruct our insight into the epistemological nature of a law. (BLA vol. I, §0)

It is thus of the utmost importance that the logical proofs of the laws of arithmetic be stated in a presuppositionless language.

¹⁵ One might think that this imperfection is not avoided in Frege's *Begriffsschrift* since, as I have indicated above, it seems that whether or not the second-level course-of-values function name has *Bedeutung* depends on the truth of a sentence: every function has a course-of-values. But this is not quite the imperfection mentioned here, Frege's description only applies to proper names. Moreover, the situation is very different in this case. As I will argue shortly, one lesson of the contradiction seems to be that 'every function has a course-of-values' is not a meaningful sentence.

Hence the "first purpose" of the Begriffsschrift, as Frege says when he introduces it, is to prevent presuppositions from sneaking in unnoticed (BEG, p. 6/ BS x). The Begriffsschrift expression of each of his propositions, Frege says in *Basic Laws*, explicitly expresses "all of the conditions necessary to its validity"; there can be no "tacit attachment of presuppositions in thought" (BLA vol. I, p. vi). This explains the nature of Frege's worry in "On Sense and Meaning". It is, as Resnik claims, a problem if a logical language contains proper names that do not have *Bedeutung*. For a logically perfect language must not contain proper names with no *Bedeutung*. But the problem that Frege finds especially worrisome is that it might contain proper names whose having *Bedeutung* depends on the truth of a thought.¹⁶ The discussion from "On Sense and Meaning" ends with the claim that, after the introduction of a logically perfect language:

Then such objections as the one discussed above would become impossible, *because it could never depend upon the truth of a thought whether a proper name had meaning*. ((Frege, 1892a), p. 41, my emphasis)

This does not, in itself, undermine the metatheory-reading. For Frege's concern here is not to prohibit presuppositions of the sort expressed in metatheoretic statements. For example, Frege does not mean to prohibit the presupposition that a simple proper name designates something. Indeed, he writes,

If anything is asserted there is always an obvious presupposition that the simple or compound proper names used have meaning (*Bedeutung*). If therefore one asserts 'Kepler' died in misery,' there is a presupposition that the name 'Kepler' designates something; but it does not follow that the sense of the sentence 'Kepler died in misery' contains the thought that the name 'Kepler' designates something. ((Frege, 1892a), p. 40)

There are, however, no such proper names in the Begriffsschrift of *Basic Laws*. These proper names are all complex. And the presuppositions that worry Frege, as his example indicates, are presuppositions that complex proper names have *Bedeutung*. Moreover, as we shall see, the sort of thoughts that worry Frege here – thoughts on which a complex proper name's having *Bedeutung* might depend --are not metatheoretic at all. The problem for the metatheory-reading arises, oddly enough, because Frege's worry about course-of-values names is not really a metatheoretical worry.

¹⁶ It is evident, from the context in "On Sense and Meaning", that the sort of proper names under discussion are descriptions rather than actual proper names. Frege's worry is not about 'Kepler', which he also discusses, but about 'whoever discovered the elliptic form of the planetary orbits'. In Begriffsschrift, however, there are no primitive proper names. Since all primitive Begriffsschrift names are function names, every proper name is formed from function names. That is, every Begriffsschrift proper name is, in effect, a description.

III. "On Sense and Meaning" and Course-of-values Names

If we take these remarks seriously, we have a partial explanation of one of the mysteries of §31. As we saw earlier, it seems mysterious that Frege does not appeal to the claim that each function has a (unique) course-of-values in his attempt to justify his claim that his second-level course-of-values function sign has *Bedeutung*. As the above discussion indicates, Frege thinks that whether or not a proper name of *Begriffsschrift* designates something cannot depend on the truth of a thought. In particular, then, whether or not a proper name of the form ' $\overset{\uparrow}{\varepsilon} f(\varepsilon)$ '—that is, a name formed using the course-of-values function sign—designates something cannot depend on the truth of the claim that each function has a (unique) course-of-values.¹⁷

But there are respects in which this explanation is unsatisfying. One of these is that it may seem that Frege is simply wrong: whether or not the proper names in question designate objects *does* depend on the truth of the thought that each function has a (unique) course-of-values. In fact, however, this is far from obvious.

It helps to note, first, that we would not be inclined to say that whether or not a proper name of the form 'the gobbledygook of $f(x)$ ' has *Bedeutung* depends on the truth of a thought: that each function has a gobbledygook. Or, at least, we would not be inclined to say this without first being told what it is to be a gobbledygook of a function. For, unless we have assigned an appropriate meaning to 'gobbledygook', the string of words 'each function has a gobbledygook' is not a sentence that has a truth-value. It is, rather, exactly the sort of nonsense it appears to be. But now consider the expression:

each function has a course-of-values.

Although 'course-of-values' is not known as a nonsense expression, it is also not a familiar expression. What reason have we for thinking that the string of words set off above expresses a thought? The answer has to come from an account of the meaning of the term 'course-of-values'. Frege's introduction of the notion of course-of-values in §3 *Basic Laws* is (BLA vol. I, §3):

I use the words

"the function $\phi(\xi)$ has the same *course-of-values* as the function $\psi(\xi)$ "
generally to denote the same as [*als gleichbedeutend mit*] the words

"the functions $\phi(\xi)$ and $\psi(\xi)$ have always the same value for the same argument".

This equivalence, when stated in *Begriffsschrift*, is Frege's Basic Law V. It also is, as he acknowledges in §10, insufficient to fix the sense of the term 'course-of-values'. But

¹⁷ Note that this claim does not belong to the metatheory. It can easily be expressed in *Begriffsschrift* using no symbols other than those introduced in *Basic Laws*.

even in §10, which is titled "the course-of-values of a function more exactly specified", he does not attempt to explain what he means by 'course-of-values'. Rather, the section is devoted almost entirely to an argument that it can be determined "for every function when it is introduced, what values it takes on for courses-of-values as arguments" (BLA vol. I, §10)

How are we to know, then, what sorts of things courses-of-values are? Although Frege says very little about this in the first volume of *Basic Laws*, in volume II he takes up the issue again.¹⁸ He begins by rehearsing the introduction from §3 of volume I:

We said: If a (first-level) function (of one argument) and another function as such as always to have the same value for the same argument, then we may say instead that the course-of-values of the first is the same as that of the second (BLA vol. 2, §146)

As before, he claims that this must be regarded as a fundamental law of logic. However, in this discussion, unlike the earlier discussions, Frege gives us some information about why this should do as an introduction of the notion of course-of-values. He indicates that he is only appealing to our antecedent understanding that this is so. The transformation of the equality holding generally into an identity, he says, is implicitly exploited all the time both by logicians and mathematicians (BLA vol. 2, §147).

But this is not a sufficient reason for not defining 'course-of-values'. After all, Frege insists on defining the concept of number, a far more familiar notion than that of course-of-values. Why does he not demand a definition of 'course-of-values'? This, Frege suggests, is simply not possible. It is not just that, as he says, the above equivalence must be regarded as a fundamental law of logic. He also says:

This transformation must not be regarded as a definition; neither the word 'same' or the *equals* sign, nor the word 'course-of-values' or a complex symbol like ' $(\varepsilon)\phi(\varepsilon)$ ', nor both together, are being defined by means of it. (BLA vol. 2, §146).¹⁹

Although there is a technical reason why the transformation must not be regarded as a definition, there is also another reason. He adds, in a footnote:

In general, we must not regard the stipulations in volume I, with regard to the primitive signs, as definitions. Only what is logically complex can be defined; what is simple can only be pointed to (*hinweisen*). (BLA vol. 2, §146).

¹⁸ Frege rarely says much about the notion of courses-of-values. When he introduces the notion of extension in *Foundations* he simply says "In this definition the sense of the expression "extension of a concept" is assumed to be known." (FA, p. 117). In the earliest introduction of the notion of course-of-values, in "Function and Concept", he simply says that if functions have the same value for each argument "here we have an equality between value-ranges (Werthverläufe)" ((Frege, 1891), p. 10). This, of course, is simply another way of describing Basic Law V. As I will argue shortly, there is a perfectly legitimate reason for this.

¹⁹ In the interest of consistency of terminology, I have substituted 'course-of-values' for 'graph' in Geach's translation of "Werthverlauf".

The second-level course-of-values function sign is, of course, one of the primitive signs in Frege's logical language. Thus the second-level course-of-values function sign cannot be defined.

How seriously should we take this answer? One might suspect that Frege is expressing an *ad hoc* view; a view introduced in the hope of avoiding awkward questions about the primitive second-level course-of-values function sign. However, that would be a mistake. The general view that primitive terms, or terms for something logically simple, cannot be defined appears throughout Frege's writings. He says this not only in the second volume of *Basic Laws* but also in the first volume and also in numerous papers published both before the first volume and after the second.²⁰

IV. Primitive Terms and Elucidation

One of the earliest of these statements is in "On Concept and Object". Frege writes that what is simple cannot be defined and that, when introducing something logically simple, the only option is to "lead the reader or hearer, by means of hints, to understand the word as is intended" (Frege, 1892b), p. 193). The topic is also taken up in §0 of volume I, where Frege explains the task of *Basic Laws*. He writes:

It will not always be possible to give a regular definition of everything, precisely because our endeavor must be to trace our way back to what is logically simple, which as such is not properly definable. I must then be satisfied with indicating what I intend by means of hints (*Winke*). (BLA vol. I, §0).

The notion of course-of-values, of course, is logically simple. And we have already seen that the sorts of hints by which Frege introduces this notion include no account of the nature of courses-of-values. We are given only some simple logical truths about courses-of-values. It seems to be Frege's view that it is not possible to state what it is to be a course-of-values.

One upshot is that we can see why Frege might think that his Begriffsschrift, although flawed, is not open to the sort of objection he discussed in "On Sense and Meaning". As we have seen, Frege claims that a logically perfect language will not be subject to such objections. For whether or not a proper name of a logically perfect language has Bedeutung cannot depend on the truth of a thought. But why should Frege think this? To answer this, consider the claim that proper names formed using the course-of-values function sign do have Bedeutung. Are there thoughts upon whose truth this claim depends? It would seem that the only candidates for such thoughts are those

²⁰ See, for example, "On Concept and Object" (Frege, 1892b), p. 195; "What is a function?" (Frege, 1904), p. 665; "On the Foundations of Geometry: Second Series" (Frege, 1906), pp. 301-306.

expressed by statements of what it is to be a course-of-values. And there can be no such thoughts.

But here, again, one might suspect that—even if this is Frege's view—it is simply wrong. It is true that he does not supply an account of the nature of courses-of-values that can be viewed as expressing a thought. And, consequently, he offers no thought of this sort whose truth is required for certain proper names to have *Bedeutung*. However, one might suspect that there is a thought of another sort whose truth is required. One might think that Basic Law V is such a thought. After all, Frege introduces his course-of-values function sign with a natural language expression of Basic Law V. And one might think that the result of Russell's proof of the contradiction also provides a demonstration that this thought is false.

But is there such a thought? It seems more accurate to say that Frege simply did not succeed, in *Basic Laws*, in assigning fixed content either to the (invented) natural language expression 'course-of-values' or to the Begriffsschrift second-level course-of-values function sign. And this seems to have been Frege's considered, post-contradiction view.²¹ If so, the natural language expression that Frege translates into Begriffsschrift as Basic Law V does not express a thought at all. Whether or not a Begriffsschrift course-of-values name—that is, a proper name of the form ' $\overset{1}{\epsilon} f(\epsilon)$ '—has *Bedeutung* does not, then, depend on the truth of the thought that each function has a course-of-values. For there is no such thought.

We can now see why, on Frege's view, there should be no way to give the sort of metatheoretic proof that, according to Resnik, Frege wants to give—that is, a metatheoretic proof that guarantees that all primitive terms have *Bedeutung*. The problem is simply that the primitive terms are terms for what is simple. Logically simple notions, Frege maintains in "On Concept and Object" as well as in both volumes of *Basic Laws*, cannot be defined but only indicated by means of hints. Frege writes, in the second volume of *Basic Laws*,

²¹ Although, in his first response to Russell on June 22nd, he claims that Basic Law V is false, this view is not entirely consistent with the strategies for circumventing the difficulty that Frege discusses in his later letters. These strategies involve introducing new conceptions of the notions of courses-of-values and extensions (see, e.g., Frege to Russell, 23 September, 1902). Similar strategies are discussed in the afterward to *Basic Laws*. He considers, for example, regarding what he now calls 'class-names' as naming improper objects (*uneigentlichen Gegenständen*). And he considers regarding these names themselves as "pseudo-proper names (*Scheineigennamen*) which would thus in fact have no denotation (*Bedeutung*)" (GGA, p. 255). When he discusses the strategy that he proposes to use for solving the difficulty, he notes that "this simply does away with extensions of concepts in the received sense of the term" (GGA, pp. 260-261).

In general, we must not regard the stipulations in Vol. i., with regard to the primitive signs as definitions. Only what is logically complex can be defined; what is simple can only be pointed to (*hinweisen*). (BLA vol. II, §146).

Moreover, in the discussions of primitive terms, Frege is not simply talking about primitive Begriffsschrift terms. Frege's natural language expression 'course-of-values', as well as the second-level course-of-values function sign, is supposed to be a term for something logically simple and hence something that cannot be defined. Nor is the problem restricted to the simple notions of logic. Although Frege's general project is to identify the foundations of arithmetic, he believes that such foundational enterprises are important for other sciences as well. One of the tasks involved in identifying the foundations of a science is to identify the primitive notions from which all its other concepts can be defined.

In a series of papers written shortly after the second volume of *Basic Laws*, Frege discusses a different science, geometry. Here, also, he considers the issue of what is necessary to introduce the primitive terms of the science. In any science, he claims, we need to make sure that we have the same understanding of our primitive terms. Since definition is not possible, the understanding is to be reached by figurative modes of expression, which he now calls 'elucidations'. The task of these elucidations is to "make sure that all who use them henceforth also associate the same sense with the elucidated word" ((Frege, 1906), p. 302). But because of the nature of elucidation, as Frege also acknowledges, there is no guarantee that it will be successful. He says "we must have confidence that such an understanding can be reached through elucidation, although theoretically the contrary is not excluded." ((Frege, 1906), p. 301)²². And, "we must be able to count on a little goodwill and cooperative understanding, even guesswork" ((Frege, 1906), p. 301).

Of course, the originator of the elucidation must, Frege claims, "know for certain what he means" and "remain in agreement with himself" ((Frege, 1906), p. 301). But Frege does not say how this certainty is to be achieved. And the evidence, as Frege surely knew when he wrote these words, is that one can go wrong even here. For Frege had already realized that he had gone wrong in his attempt to achieve this certainty with respect to the notion of course-of-values. Moreover, it is evident that, even when he was writing the first volume of *Basic Laws*, Frege was unconvinced that the hints by which he introduced his notion of course-of-values were sufficient.

²² I have altered Kluge's translation here by using "elucidation", rather than "explication", as the translation of "Erläuterung".

V. Elucidation and Section 31

What does all this tell us about the official project of § 31: the attempt to show that “Our simple names denote something (*bedeuten etwas*)”? Given this understanding of Frege’s introduction of the primitive terms of Begriffsschrift, there can be no proofs that the simple names designate anything. For the stipulations by which these terms are assigned meaning must be regarded as hints. And such hints cannot provide any guarantee that these terms do designate anything. Nor can hints figure as premises of proofs that these terms designate something. The moral, it seems, is that the arguments and discussions of § 31 must have the status of elucidation rather than proof.

This is not the only reason for thinking that § 31 must have the status of elucidation. Frege’s statements about the notions of concept and function give us another reason.

Concepts and functions are unsaturated. He writes,

For not all the parts of a thought can be complete; at least one must be 'unsaturated' or predicative; otherwise they would not hold together. ((Frege, 1892), p. 205)

One consequence is that what can be said about objects cannot be said about concepts and functions.²³ In particular, as Frege writes in a letter to Russell, “We cannot properly say of a concept name that it means something (*dass er etwas bedeute*)” (Frege to Russell, 29 June 1902). And he continues “strictly speaking, the expression ‘the meaning of a function name’ (*die Bedeutung eines Funktionsnamens*) must not be used”. But Frege’s simple names are all function names. And the something that a name denotes is its *Bedeutung*.²⁴ It follows, then, that we cannot properly say, of these names that they denote something. That is, the title of § 31 “our simple names denote something (*bedeuten etwas*)” cannot be a statement of a literal truth to be proved in that section.

It may seem, at first, as if Frege must simply have made a mistake. Surely, one might think, Frege would not say that his simple names denote something unless he meant us to take this statement as expressing a literal truth. Yet Frege often does write

²³ It would take us too far afield to explain why this, and some of Frege’s other peculiar statements about concepts and functions, are consequences of his understanding of concepts and functions as unsaturated. It is worth mentioning, however, that many philosophers think that these statements are both deeply problematic and easily separated from Frege’s important views. On their view, these statements should be regarded simply as mistakes on Frege’s part. I have argued elsewhere ((Weiner, 1990), chapter 5) and ((Weiner, 1999), Chapter 6) that these statements are integral parts of Frege’s overall view and also that, understood correctly, they are not problematic at all.

²⁴ Frege’s understanding of the notion of function creates another difficulty for the metatheory reading. Because, on Frege’s view, what can be said of objects cannot be said of functions, we cannot say that objects and functions are the entities to which linguistic (or Begriffsschrift) expressions refer. That is, there cannot be a ‘refers-to’ relation that holds between proper names and objects and that holds between function-names and functions. Although this may seem to be a trivial problem for the metatheory reading, it is not. For a discussion of the significance of this issue, see (Weiner, 1995a) and (Weiner, 1995b).

sentences that he does not mean us to take as expressing literal truths. Although he does not explicitly say that the title of § 31 is one of these, he does say, of other sentences that appear in his writings, that they cannot be regarded as statements of literal truths. He writes, in “On Concept and Object”

By a kind of necessity of language, my expressions, taken literally, sometimes miss my thought; I mention an object, when what I intend is a concept. I fully realize that in such cases I was relying upon a reader who would be ready to meet me half-way -- who does not begrudge a pinch of salt. ((Frege, 1892), p. 204)

The expressions that, taken literally, miss his thought are some of the expressions (for example “concepts are unsaturated”) that Frege uses to introduce his notion of concept. The comment about relying on the reader who will meet him halfway is echoed in a statement from a later series of papers, where he writes,

[F]or the decomposition into a saturated and an unsaturated part must be considered a logically primitive phenomenon which must simply be accepted and cannot be reduced to something simpler. I am well aware that expressions like 'saturated' and 'unsaturated' are metaphorical and only serve to indicate (*hinzuweisen*) what is meant—whereby one must always count on the cooperative understanding of the reader. ((Frege, 1903), pp. 371-72)

Later in the same series, Frege says that, for elucidation to be successful, goodwill and cooperative understanding, even guesswork may be required. The expressions in question are meant to play an elucidatory role. And it is evident from Frege’s remarks that this role can be played successfully even by expressions that cannot be regarded as statements of literal truths. Can the same be said of the title and many of the sentences of § 31?

There is an obvious difference between the statements that appear in the arguments of § 31 and the remarks about the nature of concepts that appear in Frege’s papers. The latter are attempts to introduce new terms; they appear in informal discussions and, when they appear, Frege hastens to disavow them as statements of literal truths. The arguments of § 31, in contrast, are not attempts to introduce new terms. Moreover, these arguments look like actual proofs. And, while it is a consequence of some of Frege’s remarks that the sentences of §31 do not express literal truths, this is not something that Frege explicitly acknowledges in §31. One might infer that this shows that Frege did not notice the consequence—that Frege regarded himself as proving literal truths in §31. But this inference is not warranted. To see why, it will help to consider Frege’s notion of elucidation, as well as some of his actual elucidations, in more detail.

First, the introduction of primitive terms is not the only purpose for which elucidation can be required. Frege also characterizes elucidation as something belonging to the propaedeutic of a science and something that helps in "the communication of

science to others" (1906 p. 301). And, second, Frege seems to have regarded as elucidatory some of the sections of *Basic Laws* that appear to contain actual proofs. Frege divides the sections in part 2 of volume I of *Basic Laws* into those headed "Analysis" (*Zerlegung*) and those headed "Construction" (*Aufbau*). The construction sections are the sections that, Frege claims, contain his actual proofs—these proofs are expressed entirely in Begriffsschrift; they contain no German words at all. The Analysis sections have a different purpose. They "are meant to facilitate understanding by providing rough preliminary outlines of the proofs that follow them" (BLA vol. I, p. v). The role described fits the above characterization of elucidation. Other remarks in *Basic Laws* also indicate that he regards these sections as elucidatory.²⁵

Of course, the statements in the Analysis sections do not miss Frege's thought. They do seem to be statements of literal truths. Indeed, many of them are simply natural language statements of logical laws. But this is no reason to deny that they are meant as elucidations. For Frege's elucidations are by no means limited to expressions that miss his thought. In fact, even his discussions of the notion of concept include statements of literal truths. It is part of Frege's understanding of concepts as functions that each concept holds or not of each object. And this claim, far from missing his thought, being false or being nonsensical, is a natural language statement of a logical law. In a logical language of the sort Frege introduced, it can be expressed using the following sort of formula:

$$(F)(x)(F(x) \vee \sim F(x)).$$

This statement is elucidatory, insofar as it is used to communicate Frege's understanding of the term 'concept'. But it is also a natural language statement of a literal truth of the logical theory. It is also a statement that can be expressed in Begriffsschrift. Similarly, as we saw earlier, Frege uses what he takes to be a natural language statement of a literal truth of the logical theory (Basic Law V) in his introduction of the notion of course-of-values. We cannot, then, recognize a statement as elucidatory from its content. Nor can we disqualify a discussion as elucidatory because it appears to set out a proof.

Elucidation is recognizable by the role it plays in a science.

²⁵ In particular, Frege says in footnote to section 2 that the symbol " ξ " will not occur at all in the development of the Begriffsschrift itself but "I shall use it only in the exposition of it [*Darlegung der Begriffsschrift*], and in elucidations." This symbol does occur in the Analysis sections. Since part I is labeled "Exposition of the Begriffsschrift" and the Analysis sections appear in part II, which is labeled "Proofs of the Basic Laws of Number", it seems reasonable to assume that Frege regards the Analysis sections as providing elucidations.

The analysis sections are not the only examples of elucidation that takes the form of proof. Frege explicitly identifies another discussion in *Basic Laws* as elucidatory. And this discussion, which appears in §34, has almost exactly the character of §31. In §34 Frege introduces a definition—the definition of his application-function sign. Immediately after introducing this definition, he says "although the definiens contains only familiar notation, a few elucidations (*einige Erläuterungen*) are in order" (BLA vol. I, §34).²⁶ What follows is an elaborate argument that the function just defined has *Bedeutung*—an argument very like the argument in §31. It is, indeed, just the sort of discussion that is typically identified, on the metatheory-reading, as a metatheoretic proof. Frege considers what is named by the complex object-names that, according to his definition, result from inserting proper names in the argument spaces of the function-expression. He divides the possibilities into cases and argues that, for each case, the resulting object-expression will have a *Bedeutung*. The next section begins with the words:

Here we see confirmed, what we could gather from considerations already set out, that the function-name " $\xi \cap \zeta$ " has a denotation (*eine Bedeutung hat*). (BLA vol. I, §35).

As with the §31 arguments, the apparent conclusion of the §34 argument—that a particular function-name has a *Bedeutung*—is, according to Frege's remarks about the nature of functions, strictly speaking not something that can properly be said. But, as Frege's next sentence indicates, he does not in fact mean us to take his elucidation as a proof whose conclusion is that the function name in question has a *Bedeutung*. For he continues:

Only this is fundamental to the forthcoming conduct of proofs; our elucidation could be wrong in other respects without placing the correctness of those proofs in question; for only the definition itself the foundation for this edifice. (BLA vol. I, §35).

The §34 argument, then, although it cannot be correct (because of the sort of claim that is made about a function) *need not* be correct. It is elucidatory. It is designed to get Frege's readers to understand his application-function sign.

What does this tell us about §31? It is clear from a comparison of the arguments in §34 and §31 that nothing about the way §31 is written disqualifies it from playing an elucidatory role. Just as the arguments in §31 look like proofs from our perspective, the arguments in §34 look like proofs from our perspective. This also explains some of the remarks in the preface to *Basic Laws* that are surprising if we assume that §31 is meant to include proofs. Frege suggests, in the preface, that the reader skip some sections on a first reading. For, he says, "Some matters had to be taken up in order to meet all

²⁶ I have slightly altered the translation here. Furth's translation is "a few explanatory remarks".

objections, but are nevertheless inessential to an understanding the propositions of Begriffsschrift" (BLA vol. I, p. xii). §31 is among the sections that should be skipped for this reason. But he also suggests that the reader return to the sections after working through the book the first time. The reason, he adds, is that "In this way, I believe, the suspicion that may first be aroused by my innovations will gradually be dispelled [*So wird, glaube ich, das Misstrauen allmählich schwinden*]" (BLA vol. I, p. xii). This is odd as a characterization of what is to be accomplished by proof—it is much too weak. But it is exactly the sort of characterization one would expect for sections that contain elucidations, which, on Frege's view, offer us no guarantees.

I claimed, earlier, that one result of a careful examination of §31 will be to undermine the metatheory reading. Nonetheless, there are many features of the metatheory reading with which I am in complete agreement. There is no question that, in the discursive sections of *Basic Laws*, Frege is writing about his language, Begriffsschrift—and no question that he is writing about its terms' having Bedeutung. Indeed, there is no question that Frege's aim, in these sections, is to convince his readers that his logical language does what it is supposed to do—in particular, that the language will not leave us with expressions that do not designate anything. The problems with the metatheory-reading are problems with the status it accords the discussions in these sections. One problem is the insistence that what is required, on Frege's view, is actual proof. Another is the insistence that these proofs have a particular character. I have argued that what is required, on Frege's view, is not proof but elucidation. Above, I have concentrated on the first of these problems. I will turn, next, to the second.

VI. Section 31 and Circularity

My argument, so far, comes largely from an examination of Frege's general views. It is a consequence of his views about primitive terms that there are no premises to be used in proofs that the primitive Begriffsschrift terms have Bedeutung. It is a consequence of his views about the nature of functions that the apparent conclusions of such proofs are (as Frege says to Russell) imprecise expressions. It is not consistent with these views to take §31 as providing proofs that the primitive Begriffsschrift terms have Bedeutung. There is, however, another option. The purpose of §31 may be elucidatory. And there is nothing in Frege's account of the notion of elucidation that prevents us from taking the arguments of §31 as elucidatory.

But to say that §31 is meant to be elucidatory does not yet tell us how the elucidations of § 31 are supposed to work or why Frege might have thought his readers would find them convincing. One of the problems that I have not yet addressed, the

problem about the apparent circularity of the §31 arguments, is particularly vexing. This problem seems so obviously to undermine Frege's argument—whether or not it is regarded as elucidatory—that it is difficult to see how Frege could have thought that anyone would be convinced. Even if, as Frege claims, flaws in an elucidatory argument are acceptable, it is not as if any words will do, provided they are labeled 'elucidatory'.

I will try to show next that it is a mistake to regard Frege as offering an obviously bad or circular argument here. It may seem, however, that to engage in this sort of activity is to abandon the force of regarding the argument as elucidatory. If there is a perfectly good argument in §31, why not call the argument a metatheoretic proof? One answer is that Frege's elucidatory arguments cannot count as proofs because each is based on some sort of improper statement. But, as I will try to show below, there is also another answer; an answer that has to do, not with elucidation, but with the notion of metatheoretic proof. The metatheory-reading does not tell us simply that Frege is offering proofs about the functioning of his logical language. The metatheory-reading tells us that Frege is giving proofs of a certain sort. And one of the problems with the metatheory-reading is that, even if we regard Frege as giving natural language proofs about the functioning of his logical language, the proofs in question are not the proofs that, according to the metatheory-reading, Frege needs to give.

Let return to the apparent circularity of §31. In §29, titled "When does a name denote something (*Wann bedeutet ein Name etwas?*)", Frege gives a six-clause answer (one clause for each type of Begriffsschrift expression). He tells us, however, that this answer should not be taken as an account of what it is to have *Bedeutung*. For, as he says in the next section, the application of this six-clause explanation "always presupposes that we have already recognized some names as denoting (*als bedeutungsvolle*)" (BLA vol. I, §30). Such a claim seems perfectly reasonable, of course, if the explanation offered in §29 is designed to be used in the induction step of an inductive proof. For it is in the nature of an induction step to make use of such presuppositions. And most philosophers who have written about this part of *Basic Laws* agree that §29 is part of an inductive proof. Resnik writes that the clauses of §29

are to be used to show that names formed from names already known to denote also denote. Plainly, on pain of an infinite regress, the referentiality of some names must be established by other means..... Section 31 then opens with the remark that given what has been said before, one need only show that the simple names of the system denote in order to show all the names of the system do, and then turns to the proof that the simple names do denote..... To clinch the case, remember that simple names, with the notable exception of the abstraction function name, are directly assigned references. ((Resnik, 1986), p. 184)

On this interpretation, §29 and §30 are both parts of the inductive step of the proof—the clauses of §29 are to be used only in §30. In §31, Resnik suggests, Frege must use other means to deal with the basis case; to show that "our simple names denote something (*bedeuten etwas*)".

The problem with this interpretation is that Frege does not do what he is supposed to do in §31. That is, Frege does not establish the referentiality of simple names by other means. Although basis cases of inductive proofs typically appear first, it is no accident that §31 does not appear before §29. This is, of course, explicable if, in addition to the straightforward arguments that the non-problematic primitive terms have *Bedeutung*, Frege also offers a second inductive proof in §31—a proof by induction that his second-level course-of-values function sign has *Bedeutung*. And the consensus, in the secondary literature, is that he does. But there is something else that is not explicable. Frege's employment of the §29 criteria is not limited to this purported inductive proof. He employs these criteria even in his discussion of the horizontal. For he says that its having *Bedeutung* can be established by considering expressions that result from putting a proper name in its argument place. He makes similar statements in his characterizations of what must be shown for each of the other primitive *Begriffsschrift* signs. Some of the arguments are trivial and some are elaborate, but in each case there is no question that Frege is making use of the criteria introduced in §29. And it is implausible to suppose that he employed the §29 criteria in §31 without noticing that he was doing so. Yet, as we saw earlier, and Resnik indicates, the proof will be circular unless some parts of the §31 proof are accomplished without making use of the §29 criteria.

There are, I think, two keys to making sense of the arguments in §31. One, as I have already indicated, is that Frege means §31 to be elucidatory. The second is that Frege does not mean §31 to set out the basis case of inductive argument. It should be evident from my earlier discussion of elucidation that these two claims are independent. Given that the natural language statements of mathematical arguments in the analysis sections of *Basic Laws* are to be regarded as elucidatory, it is perfectly plausible that Frege might decide to give, in §§29-31, an elucidation that takes the form of an inductive argument. But there are important reasons for thinking that he did not regard §31 as part of an inductive argument—whether elucidatory or not. For Frege employs the §29 criteria in each of these arguments, and he says that the employment of these criteria "always presupposes that we have already recognized some names as denoting (*als bedeutungsvolle*)" (BLA vol. I, §30). Unless Frege has forgotten this statement from the previous section, he must regard the §31 arguments as resting on the presupposition that we have already recognized some names as denoting.

Moreover, Frege clearly does seem to be intentionally making this sort of presupposition in §31.²⁷ For, after listing all the primitive Begriffsschrift names but before discussing them individually, he writes:

We start from the fact that the names of truth-values denote something, namely, either the True or the False." (BLA vol. I, § 31).

The names in question, of course, cannot be primitive Begriffsschrift names—for the primitive names are function names and truth-values are objects. Truth-value names are complex expressions that contain the primitive names. If Frege is, as he claims, presupposing this, then §31 clearly is not meant to be setting out the basis case of an inductive argument. But what is he doing, and why does he think he can presuppose this?

Let us begin with the presupposition. The application of the §29 criteria requires us to presuppose, he says, that "*we have already recognized* some names as denoting [*das man einige Namen schon als bedeutungsvolle erkannt habe*]" (BLA vol. I, §30 emphasis added). This is not a presupposition about the symbols but about his readers. It is important, also, to note that that the application of the §29 criteria does not require that we start with names that have been *proved* to have Bedeutung but, rather with names that have been *recognized* as having Bedeutung [*der als bedeutungsvoll anzuerkennenden*] (BLA vol. I, §31). Why does Frege think that he can presuppose that his readers have already recognized some names as having Bedeutung? The answer is to be found by looking at the earlier sections of *Basic Laws*. We should have already recognized some truth-value names as having Bedeutung because Frege has already shown us this.

One of these names is ' $a \rightarrow (b \rightarrow a)$ '.²⁸ Frege has already tried to convince us that its Bedeutung is the True. Frege says in §18,

By §12,

$$\Gamma \rightarrow (\Delta \rightarrow \Gamma)$$

could be the False only if both Γ and Δ were the True while Γ was not the True. This is impossible; therefore

$$\vdash a \rightarrow (b \rightarrow a) \quad (I)$$

(BLA vol. I, §18).

The name in question, of course, is also an expression of Basic Law I. On the interpretation I am offering, this argument must be viewed as elucidatory. But this is not

²⁷ Christian Thiel notes this in (Thiel, 1965), p. 80, but suggests that the presupposition is warranted by earlier proofs.

²⁸ This is a bit oversimplified. Given the way Frege's notation works, both a and b are universally quantified. A more revealing contemporary representation of Frege's law might be:

:

$$(a)(b)(a \rightarrow (b \rightarrow a)).$$

The difference is not significant for our purposes here.

to say that it is in any way problematic. It seems reasonable for Frege to suppose that his readers, having read §18, recognize that the expression ' $a \rightarrow (b \rightarrow a)$ ' is a truth-value name. Frege provides similar arguments for each basic law. Thus it seems perfectly reasonable for Frege to assume that, by the time his readers reach §31, they will recognize each Begriffsschrift expression of a basic law as a name of a truth value (the True). In the argument in §31, as I interpret it, Frege means to exploit this assumption. To see the significance of this interpretation, it will help to contrast it with the metatheory-reading.

The fact from which we start in §31, Frege says, is that the names of truth-values denote something. It is evident that Frege does not mean that truth-value names—*if there are any*—have Bedeutung. For he says, next, that we then gradually widen the sphere of names to be recognized as having Bedeutung. Talk of widening this sphere does not make sense unless we begin with names that we recognize as having Bedeutung. But it is difficult to see how, on the metatheory-reading, the proofs of §31 could start with this fact. The only Begriffsschrift candidates for truth-value names are complex. Since, on the metatheory-reading, compositionality is a central feature of Frege's theory of reference, a truth-value name can have Bedeutung only if all its constituent expressions have Bedeutung. Thus to start with this fact is to presuppose that at least some simple function names do have Bedeutung.²⁹ Yet § 31 includes, for each primitive name, an argument that it has Bedeutung.³⁰

²⁹ In (Heck 1997), Richard Heck offers a different account of this part of section 31. Heck asks what Frege's assumption (that the names of truth-value denote something) amounts to and answers:

I see no option but to suppose that what Frege means is that, if an expression denotes a truth-value, then it denotes *something*. So the assumption amounts to the stipulation that the domain of the theory is not empty, that, in particular, it contains the two truth values. (Heck 1997, p. 449).

This much is certainly true: Frege does assume that the two truth values are among the things over which his quantifiers range (I resist talking of a domain here, for reasons described later in this paper). But this does not exhaust the content of Frege's assumption. Frege's assumption, as stated, is an assumption about *names*.

What are these names? As I argue below, it is perfectly reasonable for Frege to assume in §31 that we have already recognized some Begriffsschrift expressions as names of truth-values. The Begriffsschrift statements of basic laws have the status of complex Begriffsschrift names and one upshot of Frege's discussions should be to convince us that they are names of the True.

On Heck's interpretation, however, Frege's section 31 assumption is not an assumption about Begriffsschrift names. Heck argues that we should not understand Frege to be talking about filling in argument-places with actual expressions but, rather, with what Heck terms "auxiliary expressions". He writes:

The term ' Δ ' is not supposed to be a name in Begriffsschrift *at all*: it is a formal device, a *new* name, added to the language, subject only to the condition that it should denote some object in the domain. (Heck 1997, p. 442).

The passage from §18 quoted above also presents another problem for the metatheory-reading. On the metatheory-reading, this passage must set out a

Although I do not have room to discuss all the difficulties with this interpretation, it will be useful to mention, briefly, its significance for the passages that I have been discussing in this paper. Frege begins section 30 by saying that the application of the section 29 provisions presupposes ‘that we have already recognized some names as denoting’. On Heck’s interpretation, however, this is misleading. For while Frege thinks that we *can* apply the section 29 provisions, he does not think that we really have already recognized actual names as denoting. What Frege should have said, on Heck’s interpretation, is that we can apply the section 29 criteria because we recognize that we can stipulate that a capital Greek letter is to denote something and add it to Begriffsschrift. Frege’s next comment—that the section 29 provisions can only be used in the extension of the sphere of names we recognize as denoting—is also misleading. The auxiliary-name technique allows us to start with a situation in which *no* names are recognized as already denoting. We then use the section 29 provisions to show that, because we can assign auxiliary symbols a denotation, we can recognize actual names as already denoting. Frege is, given Heck’s interpretation, similarly misleading us when he goes on, in section 31, to say that we start with the fact that ‘the names of truth-values’ denote something and that we then “widen the sphere of names to be recognized as succeeding in denoting”. Again, on Heck’s view, there is no sphere of names that are already recognized as denoting. Instead, on Heck’s view, Frege means to be saying that we start with the fact that, should we stipulate that ‘ Δ ’ denotes a truth-value, given that there *are* truth-values, ‘ Δ ’ denotes something. There is no sphere of already recognized-denoting names to widen. Why should Frege want to mislead his readers in this way? It is not particularly difficult to express the views that, according to Heck, Frege actually holds. What Frege *ought* to say, were Heck’s interpretation correct, is: we start from the fact that *it is possible* to add non-Begriffsschrift proper names that denote truth values and, from this, we can show that *actual* Begriffsschrift expressions have Bedeutung. It is plausible that someone might hold the sort of view Heck attributes to Frege. But there is no textual evidence that Frege held it. It is very odd that Frege should have adopted this elaborate and ingenious strategy to make his proofs work yet never explained it to his readers. In particular, no such discussion appears in Frege’s explanation of his use of the capital Greek letters in §5 of *Basic Laws*.

Moreover, if Heck is right, then capital Greek letters are misused in the passage from §18 quoted above. On Heck’s interpretation, Frege uses capital Greek letters, not as metalinguistic variables but as actual symbols that can be assigned a denotation and added to Begriffsschrift. Thus in §18 Frege means to be saying that, whatever values are assigned to ‘ Γ ’ and ‘ Δ ’, ‘ $\Gamma \rightarrow (\Delta \rightarrow \Gamma)$ ’ names the True. He could have made this clear in the passage quoted by writing either

if both ‘ Γ ’ and ‘ Δ ’ named the True

or

if both ‘ Γ ’ and ‘ Δ ’ were assigned the True

Instead, Frege writes something that makes no sense at all, on Heck’s view. He writes,

if both Γ and Δ were the True while Γ was not the True.

Such uses of capital Greek letter, which are improper according to Heck’s interpretation, appear throughout the early sections of *Basic Laws*. Again, given the ease with which Frege could have done this correctly, had he held the views that Heck attributes to him, it is difficult to see why these sections are written as they are.

³⁰ One might argue that the reason is that Frege thinks he has already shown that some of his primitive names have Bedeutung. Most of the arguments in section 31, after all, are fairly perfunctory. But this will not help. For the most perfunctory arguments are for the horizontal and negation stroke—he simply says, “This follows immediately from our explanations” (BLA vol. I, §31). Yet he precedes this statement with the claim that we need to show that putting a name of a truth-value in the argument place of these function names results in names that are *bedeutungsvoll*. That is, we are supposed to make use of the assumption that there are Begriffsschrift names of truth-values even for the purpose of arguing (however perfunctorily) that the horizontal and negation stroke have Bedeutung.

metatheoretic proof. But it is difficult to see exactly what it is that the proof is supposed to show. It cannot be an attempt to offer a metatheoretic proof that shows—not that Basic Law I is true—but that the Begriffsschrift expression has the property of naming the True under Frege’s interpretation (or of expressing a true thought on Frege’s interpretation). For the passage contains no reference to Begriffsschrift expressions at all. It contains no statement about the truth of thoughts or sentences. Nor does it contain a truth predicate. The predicate ‘names the True’, which might plausibly be taken as a truth predicate, appears nowhere in §18. The only predicate that appears in the passage and that one might take to be a truth predicate is the predicate ‘is the True’. And it is a mistake to take this to be a truth predicate. For the True is an object and the ‘is’ in the predicate ‘is the True’ is the ‘is’ of identity. True is neither a sentence nor a thought. Hence to say, of a sentence or thought, that it is the True is to say something false.

This may seem too quick. After all, the True is the extension of a concept and, given that numbers are defined as extensions of concepts, there is no obvious reason why sentences and thoughts might not similarly be defined as extensions of concepts. A moment’s reflection shows, however, that if we take the True to be a sentence or thought, then ‘is the True’ is not a truth predicate. For, if ‘is the True’ is a truth predicate, it must hold of all true sentences. That is, each true sentence must *be* the True. Consequently, there is only one true sentence. Clearly, this will not do. And for similar reasons the True cannot be a thought.

Now Frege could certainly use, and sometimes does use, an expression that almost seems to qualify as a truth predicate: the expression "*das Wahre bedeutet*".³¹ But he does not use this expression in the passage set off above.³² Of course nothing prevents us from

³¹ It is, however, also worth noting that "*das Wahre bedeutet*" is not exactly a truth predicate in any contemporary sense. For a truth predicate, presumably, holds only of true sentences or thoughts. And this predicate holds of an expression, " $\varepsilon(\text{---}\varepsilon)$ " which is not a true sentence.

³² It is interesting, in this context, to note that, in a paper devoted to trying to show why Frege must have had a metatheory, Jason Stanley writes:

Now a central thesis of this paper is that one way in which Frege uses his truth-predicate is in giving a justification, in the metalanguage, of the rules of inference of the *Begriffsschrift*.
((Stanley, 1996), p. 47)

and he continues:

The justification of the inference rule of Modus Ponens lies in the fact that if the conditional is true, and its antecedent is true, then so is the consequent. This fact gives some justification to accepting Modus Ponens as a rule of inference in a logical system by pointing to its truth-preserving nature.

But the predicate ‘is the True’, as we have seen, is not a truth predicate. Frege’s justification includes no use of a truth predicate. The justification of Modus Ponens is (BLA vol I §14):

from the propositions " $\vdash \Gamma$ " and " $\vdash \Delta$ " we may infer " $\vdash \Gamma$ ";
 $\vdash \Delta$

rewriting the passage so that it contains recognizable statements about sentences or thoughts. And the result would seem to be a metatheoretic argument. But to show that Frege's argument can be rewritten as a metatheoretic argument, hardly provides support for the metatheory-reading. If Frege meant to make a metatheoretic argument here, why did he not do so?

In fact, it is easy enough to see what Frege means the discussion of Basic Law I to do for his readers. These readers, to whom his notation is new, may not immediately grasp the sense of the Begriffsschrift expression of Basic Law I. The statement:

$$\Gamma \rightarrow (\Delta \rightarrow \Gamma)$$

could be the False only if both Γ and Δ were the True while Γ was not the True.

looks simply like a natural language gloss on how to understand Begriffsschrift expressions of the form ' $\Gamma \rightarrow (\Delta \rightarrow \Gamma)$ '. And Frege explicitly recommends elucidation for the enterprise of showing us how to understand Begriffsschrift expressions. Moreover, once we understand the Begriffsschrift expression of Basic Law I, we can also see that it is true—without resort to any demonstration that its constituents are correlated with extra-linguistic entities. Frege is introducing a new language and these discussions help us to understand it.

One might object that this is not a plausible explanation of Frege's aim. Since, by §18, Frege has already introduced all the Begriffsschrift symbols that appear in Basic Law I, he should have no need to explain the expression—he should expect his readers already to understand the Begriffsschrift expression. But Frege typically does not expect this, nor should he. After all, Frege's §34 definition of the Begriffsschrift sign for the application function is formed from previously introduced Begriffsschrift expressions. Yet, as we saw earlier, he thinks it is also important to provide elucidation—including a natural language gloss explaining its content. It is perfectly reasonable to suppose that Frege would find it equally important to give such glosses for the Begriffsschrift expressions of his basic laws. In §18, and the other sections in which basic laws are introduced, Frege is not offering metatheoretic proofs that each Begriffsschrift expression

for if Γ were not the True, then since Δ is the True, $\neg\Gamma$ would be the False.

$\uparrow_{-\Delta}$

Stanley also writes, "In assertions of the validity of inference rules, the truth-predicate occurs *ineliminably*." (Stanley, 1996, p. 53). But, while this is certainly true in contemporary discussions of logic, as the above indicates, on Frege's conception of sentences as names of truth-values there need be no ineliminable appearance of a truth-predicate in assertions of the validity of inference rules. Indeed, as we can see from the passage set off above, in Frege's own discussion of the validity of Modus Ponens, there is no appearance of a truth-predicate at all.

of a basic law names the True. Rather, he is enabling his readers to understand what is meant by these expressions and, at the same time, to recognize each of them as naming the True.

The Begriffsschrift expressions of the basic laws are not the only proper names that Frege's readers should recognize as having Bedeutung by the time they reach §31. There are, in addition, two further names, which are discussed in §10. In §3 Frege introduced both his notion of course-of-values and a notation for the course-of-values function name. We can use the course-of-values function name to form a name for the course-of-values of a first-level function $\Phi(\xi)$. The name looks like this:

$$\overset{\uparrow}{\varepsilon} \Phi(\varepsilon)$$

Frege uses only (a natural language version of) Basic Law V to introduce these names. But the use of Basic Law V is insufficient to establish what courses-of-values are. In particular, it does not commit us to recognizing truth-values as courses-of-values; nor does it commit us to recognizing truth-values as objects that are not courses-of-values. Now, supposing that " Δ " is a name of a truth value, there is nothing that determines whether or not it is true that

$$\Delta = \overset{\uparrow}{\varepsilon} \Phi(\varepsilon).$$

But, assuming " $\overset{\uparrow}{\varepsilon} \Phi(\varepsilon)$ " does name some object, it must be determinate whether or not the object named is Δ .

In §10, Frege offers us a solution to this difficulty. Without contradicting Basic Law V, he says, "it is always possible to stipulate that an arbitrary course-of-values is to be the True and another the False." (BLA vol. I, §10). He then proceeds to stipulate that the course-of-values of the horizontal function, $\overset{\uparrow}{\varepsilon} (\text{---} \varepsilon)$, is to be the True and that another course-of-values,³³ $\overset{\uparrow}{\varepsilon} (\varepsilon = \sim(a)(a=a))$ is to be the False. The upshot is that two new Begriffsschrift names have been recognized as having Bedeutung, ' $\overset{\uparrow}{\varepsilon} (\text{---} \varepsilon)$ ' and ' $\overset{\uparrow}{\varepsilon} (\varepsilon = \sim(a)(a=a))$ '. These names, like the expressions for the basic logical laws – which we already recognize as having Bedeutung – are names of truth-values

We can now see why the arguments in §31 are not circular. Frege is using the §29 criteria and he has explained, in an earlier section, how these clauses can be used. He says:

their application always presupposes that we have already recognized some names as denoting (*als bedeutungsvolle*). They can serve only in the extension step-by-step of the sphere of such names. (BLA vol. I, § 30).

³³ This course-of-values is the extension of the concept *is identical to the False*.

The sphere to be extended, of course, is not the sphere of names that *have* Bedeutung; it is not the sphere of names that *have been proved to have* Bedeutung; it is, rather, the sphere of names that we *recognize* as having Bedeutung. In §31 Frege means to be using our antecedent recognition that the Begriffsschrift expression of the basic laws, as well as 'ε' (— ε)' and 'ξ' (ε = ~(a)(a=a))', have Bedeutung to get us to recognize that all the simple Begriffsschrift names have Bedeutung. His first step is to draw our attention to our antecedent recognition that these complex expressions have Bedeutung. All these expressions are truth-value names. Thus, after listing the primitive Begriffsschrift names, Frege writes, "We start from the fact that the names of truth-values denote something, namely, either the True or the False." (BLA vol. I, § 31). He continues:

We then gradually widen the sphere of names to be recognized as succeeding denoting (*als bedeutungsvoll anzuerkennenden Namen*) by showing that those to be adopted, together with those already adopted, form denoting names (*bedeutungsvolle Namen*) by way of the one's appearing at fitting argument-places of the other. (BLA vol. I, §31).

There is no obvious circularity in this strategy.

But if there is no obvious circularity, it may seem that nonetheless there is a problem with Frege's procedures. To see why, let us return to one of the passages quoted earlier from § 31. Frege writes,

In order now to show, first, that the function-names "— ξ" and "⊥ ξ" denote something (*etwas bedeuten*), we have only to show that those names succeed in denoting (*bedeutungsvoll sind*) that result from our putting for "ξ", the name of a truth-value (we are not yet recognizing other objects). (BLA vol. I, §31).

It should be evident that this procedure is in accord with his talk about gradually widening the sphere of names that are recognized as having Bedeutung. For the only Begriffsschrift names that have been recognized as having Bedeutung, at this point in *Basic Laws*, are truth-value names. But this should not be sufficient to satisfy the §29 criteria. Consider what the §29 criteria demand in order to show that the horizontal has Bedeutung. We must show that every proper name is bedeutungsvoll that is formed by inserting a bedeutungsvoll proper name in the argument place of the horizontal. And there are Begriffsschrift proper names (formed using the second-level course-of-values function name) that, although this has not yet been shown, *do* have Bedeutung but are not truth-value names. Frege's official procedure is, it seems, simply wrong.

The situation is not as clear as it seems, however. Frege's aim is to have shown, by the end of the argument in section 31, that all Begriffsschrift names have Bedeutung. He begins by noting (very little, after all, is required for this) that the result of inserting a truth-value name into the argument place of the horizontal is a bedeutungsvoll proper name. Only later in the section are we confronted with Begriffsschrift proper names that

are not truth-value names. And the earlier discussion of the horizontal places a requirement on any attempt to show that these names have *Bedeutung*. Thus, immediately after introducing a category of proper names that includes some non-truth-value names (the fair course-of-value names), he writes,

We must examine whether a fair course-of-values-name placed in the argument-places of " $\text{---} \xi$ " and " $\text{---} \xi$ " yields a denoting proper name (*einen bedeutungsvollen Eigennamen*)... (BLA vol. I, §31).

In the discussion that follows, he argues that the discussion in §10 allows us to see that fair course-of-values-names satisfy the requirement.

There is a sense in which this is a kind of inductive argument, for the gradual widening takes place in stages and can be described only by introducing a hierarchy of *Begriffsschrift* expressions.³⁴ On the first level are the proper names that have been recognized, as a result of earlier discussions, as having *Bedeutung*. These are truth-value names. Frege then proceeds to argue that the result of placing truth-value names in the argument place of first-level function names yields denoting proper names. The results, of course, are still truth-value names. Once these first-level functions are recognized as having *Bedeutung* (by the §29 criteria and insofar as we are limited to proper names that have been already recognized as having *Bedeutung*), Frege can proceed to his second- and third-level function signs. The quantifier comes next. The results of forming proper names of the next stage, using the quantifier function-name (a second-level function name) and first-level function names are also truth-value names. This is followed by the third level function name and, finally, the most difficult case: the second-level course-of-values function name, which itself requires a complicated argument involving a sort of hierarchy. The importance of the hierarchy is to convince us that the procedure will ultimately encompass all *Begriffsschrift* expressions. But it is also important to note that to say that Frege uses this hierarchy to structure his argument is not to say that Frege's argument is by induction on the complexity of an expression.

As we have seen, the expressions that we come to recognize, first, as having *Bedeutung*—the proper names used for basic laws and the two course-of-values names for the True and the False—are not primitive, but complex expressions. They are, in particular, more complex than the expressions that we are supposed to come to recognize, in §31, as having *Bedeutung*. It is only in the process of widening the sphere of names to be recognized as having *Bedeutung* that we come to recognize primitive *Begriffsschrift*

³⁴ Due to limitations of space, I am not able to give an account of how this argument works and, in particular, Frege's procedure in arguing that the second-level course-of-values function name has *Bedeutung*. I hope to cover this in a sequel to this paper, currently in preparation.

expressions as having *Bedeutung*. Indeed, this is merely another way of noting that Frege employs the §29 criteria in the arguments of §31. That is, he bases his arguments that the simple *Begriffsschrift* names have *Bedeutung* on the assumption that we recognize *other* *Begriffsschrift* names as having *Bedeutung*. Moreover, the §31 arguments that these primitive *Begriffsschrift* expressions have *Bedeutung* involve arguing that still other even more complex expressions do. It should be evident at this point that, if we assume that Frege is making an inductive argument on the complexity of an expression, circularity will be inevitable.³⁵

³⁵ There are, in the secondary literature, a number of other arguments that section 31 is circular, most of which concern the details of the purported inductive proof. See, for example (Parsons, 1965), (Resnik, 1986) and (Thiel, 1996). Although I do not have room to discuss these arguments here, it is instructive to see the role played in one of these discussions by the assumption that Frege is making a proof by induction on the complexity of an expression. Using an example that also appears in (Thiel, 1965), Resnik writes that following the model of the rest of Frege's proof, the way to show that an abstract A is a referential abstract is to show that "(g) (g (A))" is referential.... However that is shown by showing that $F(A)$ is referential for every referential first-level function name $F(\xi)$ and "(g) (g (ξ))" is one of these; so we have gone in a circle. ((Resnik, 1986), p. 187).

Although I am not convinced that Thiel and Resnik are right about what must be shown to show that A is a referential abstract, what is most interesting for purposes here is what Resnik says is required to show that "(g) (g (A))" has *Bedeutung*. Why should this require us to show, *first*, that $F(A)$ has *Bedeutung* for every first-level function name $F(\xi)$ that does? Supposing, as Resnik does, that in §§29-31 Frege is offering an inductive proof on the complexity of an expression that all *Begriffsschrift* names have *Bedeutung*, then this proof should be applicable to show that particular names have *Bedeutung*. And the application of this sort of inductive proof to show the expression "(g)(g(A))" has *Bedeutung* requires a demonstration that proceeds from simple signs to more complex signs. Thus the circularity Resnik finds is dependent on his regarding Frege's proof as an inductive proof on the complexity of an expression. One might think, however, that there is a problem here in any case. How can Frege show that the expressions in question have *Bedeutung*?

To find an answer, we need to look for a place in which Frege actually discusses a proper name that has a second-level quantifier. The natural place to look is in Frege's discussion of Basic Law IIa. Frege writes:

We now understand by " $(f) ((a)f(a) \rightarrow f(\Gamma))$ " the truth-value of *one's always obtaining a name of the True, whatever function-name one may substitute in place of "f" in " $(a)f(a) \rightarrow f(\Gamma)$ ". This truth-value is the True, whatever object " Γ " may denote. (BLA vol. I, §20—I have substituted some contemporary notation for Frege's here; the italicization is in the original.)*

Here we have an argument that a complex *Begriffsschrift* expression beginning with a second-level quantifier has *Bedeutung*. It is an argument in which Frege does not follow the procedure that Resnik outlines. We can easily make a similar argument that "(g) (g(A))" names the False, whatever object A is. For, using Frege's words, we understand by "(g) (g(A))" the truth-value of *one's always obtaining a name of the True, whatever function-name one may substitute in place of "g" in "g(A)".* And we know that, since one of the first-level function names is $\sim\xi=\xi$, this truth-value is the False. If, as I have argued, Frege's view is that a complex *Begriffsschrift* expression can be shown to have *Bedeutung* *without* an antecedent demonstration that its constituents have *Bedeutung*, the circularity problem, as Resnik states it, does not arise.

VII. Section 31 and Another Puzzle

The apparent circularity of section 31 is one of two problems that almost immediately bother most people when they read section 31. The other problem is also rooted in the §29 criteria. The §29 criteria consist of six clauses, one for each type of Begriffsschrift name. In each clause, we are told that a name of the type in question has Bedeutung just in case a certain group of expressions in which it appears have Bedeutung. The expressions in question are all proper names. Thus one salutary feature of employing the §29 criteria is that when Frege argues that a particular function name has Bedeutung, the substance of the argument need not actually contain the sort of talk that, he tells Russell, ought to be prohibited. He need not include talk about the meaning of a function name. Instead, he can limit his discussion to that of whether the members of a class of proper names in which the function name appears have Bedeutung.

The strategy has a less salutary —although not very surprising—feature as well. The employment of the §29 criteria will not suffice to show that a function name names a function. Frege is adamant that functions must have values for every argument of the appropriate type. The addition function, he tells us, must be defined not only for the numbers but also for the moon (BLA vol II, § 64). Consider a first-level function name. Suppose Frege shows that every result of putting a Begriffsschrift proper name in its argument place is a bedeutungsvoll proper name. This seems only to show that the function name names something that has values for objects with Begriffsschrift names. Frege's own statements seem to require that, to show that the function name names a function, one must show that it has a value on the moon. Yet there is no Begriffsschrift name for the moon.

On the metatheory-reading, this is a serious problem. For the employment of the §29 criteria cannot suffice to show that the primitive Begriffsschrift function names are associated with the appropriate sorts of extra-linguistic entities. As I have argued, it is not Frege's aim in these sections to show that Begriffsschrift function names are associated with the appropriate sorts of extra-linguistic entities. But this is not, itself, a solution to the problem. If it is not Frege's aim to show this in §§28-31, what is his aim? Frege says, in the introduction to *Basic Laws*, that the arguments of these sections are supposed to allay the suspicions aroused by his innovations. What suspicions are likely to be aroused? One obvious answer is that the reader will be suspicious of the second-level course-of-values function name—indeed, Frege himself alerts the reader to his own worry about this. Yet, as we have seen, this suspicion is not the sort that can really be

addressed by the arguments of § 31. There is, however, another suspicion that is likely to bother the reader and that can be addressed by these arguments.

Frege has argued, in earlier sections, that his expressions of logical laws express truths and that only truths follow by his rules of inference from truths.³⁶ Each line of each proof gives us a proper name preceded by a judgement stroke and horizontal. Frege needs his readers to be confident that each of these proper names names the True. The earlier arguments should convince the reader that it suffices for each of these proper names to satisfy the following condition: it must either come from earlier lines by application of a rule of inference or it must be a logical law or definition. By their construction, these names that appear in Frege's proofs seem to satisfy the condition. But there could still be a worry. Suppose one were to replace the variables of a basic law with Begriffsschrift proper names that have no Bedeutung. The result would appear, by its construction, to satisfy the condition, yet would not.

This can be avoided if every Begriffsschrift proper name has Bedeutung. Should Frege expect his readers to find it obvious that every Begriffsschrift proper name does have Bedeutung? Many of the proper names that appear in Frege's proofs are either rife with defined abbreviations or immensely complex. And the definitions are themselves immensely complex. Thus Frege may well have wanted to convince the reader that nothing goes wrong in the process of forming these complex names. And the satisfaction of the §29 criteria by all Begriffsschrift expressions is exactly what is required for this. This will show that any result of embedding Begriffsschrift expressions in more complex proper names will have Bedeutung. If Frege has succeeded in convincing the reader that the §29 criteria have been satisfied, he will not have succeeded in convincing the reader that all Begriffsschrift expressions are associated with extra-linguistic entities. But he will have convinced the reader that all lines of his proof are in order.

There may still seem to be a problem here. For the upshot of Frege's use of the §29 criteria may seem to be that, at least in some cases, we have received the wrong verdict about what the Bedeutung of an expression is. To see this, consider Frege's discussion of the universal quantifier. Frege argues that a name ' $(x)\Phi(x)$ ' has Bedeutung just in case

³⁶ It is probably worth reminding the reader that to say that Frege makes these arguments is not to say that he has a metatheory in anything like the contemporary sense. In particular, as we have seen in the above discussion of §18, where Frege discusses Basic Law I, and of §14 (32n), where Frege discusses Modus Ponens, there is no use of a truth predicate. That is, no truth predicate is required for Frege to show that Basic Law I names the True and that modus ponens is truth preserving. A more complete discussion of this issue appears in my forthcoming, "Understanding Frege's project".

' $\Phi(\xi)$ ' does.³⁷ Moreover, ' $\Phi(\xi)$ ' has Bedeutung just in case every proper name that results from inserting a Begriffsschrift proper name in its argument place has Bedeutung. Frege goes on to say, in §30, that if ' $\Phi(\xi)$ ' has Bedeutung and if every proper name that results from filling in the argument place names the True, then ' $(x)\Phi(x)$ ' names the True. And this is where Frege seems to have given us the wrong account. For it would seem that every *object* has to satisfy ' $\Phi(\xi)$ ' to make the universal generalization true. And, according to Frege's claim in §31, the universal generalization is true provided all objects that have Begriffsschrift names satisfy it. Yet not all objects have Begriffsschrift names. Moreover, Frege comes close to giving the right account elsewhere in *Basic Laws*. In §8, for example, he suggests that ' $(x)\Phi(x)$ ' names the True just in case "for every argument the value of the function $\Phi(\xi)$ is the True".

The problem with the §31 account can be made more vivid by considering the statement that every object is a course-of-values—a statement that is easily expressible in Begriffsschrift by something like: $(x) (\exists F) (x = \varepsilon F(\varepsilon))$. The problem is that each Begriffsschrift proper name formed from the symbols introduced in *Basic Laws* is a name of a course-of-values. Either it will have been formed using the second-level course-of-values function name or it will be a truth-value name, hence it will either name the True, which is $\varepsilon(\text{---}\varepsilon)$, or the False, which is $\varepsilon(\varepsilon = \sim(a)(a=a))$. Thus the Begriffsschrift statement that every object is a course-of-values names the True—it is, it seems, a logical truth that every object is a course-of-values.

There is, however, a flaw in this reasoning. For the so-called logical truth cannot be proved in Begriffsschrift. It can be shown only by a metatheoretic argument. And the metatheoretic argument depends on facts about particular objects (in this case, the logical language). Thus the so-called logical truth does not qualify as a logical truth—it cannot be shown with the sole support of those laws of thought that transcend all particulars. Of course, it need not be a logical truth to cause problems. It is a considerable problem if this is true at all. But here, again, the problem is a result of placing our faith in metatheory and on natural language. If we take Frege to be giving truth conditions for Begriffsschrift statements in which the quantifier appears we must, simply, accuse him of making a mistake. But as I have argued, it is a mistake to do this. Moreover, it is a

³⁷ It is worth noting here that, strictly speaking, ' $\Phi(\xi)$ ' is not a function-name. Function names, like functions, are unsaturated or incomplete. When Frege introduces the use of ' ξ ' here, it is after noting that the ' x ' in our usual attempts to name functions is not actually part of the function name. Rather the function name is "the part of the expression which is there over and above the ' x '" (BLA vol I §1). The ' ξ ' is used in the place of ' x ', he tells us, in the exposition of Begriffsschrift and in elucidations.

mistake to think that there are cases in which we can use natural language, but not Begriffsschrift, to evaluate a claim.

Nonetheless, it will not suffice simply to say that it is not Frege's aim to provide a metatheoretic account of the contribution the first-level quantifier makes to the truth values of Begriffsschrift expressions in which it appears. After all, we can see that there is (at least in *Basic Laws*) no Begriffsschrift name for the moon. Hence we can see, it seems, that what he says in §31 is wrong. And while Frege does say, of some of his elucidations, that they go wrong of necessity, this is not one of them. It may seem that the only way to save Frege is to interpret the logic of *Basic Laws* as ranging over a particular, restricted domain – a domain consisting of courses-of-values of logical functions with Begriffsschrift names.³⁷ This interpretation may save Frege's statement about the first-level quantifier symbol, but the price is high. There are a number of problems.

One of these is that there is no direct textual support for the interpretation. And it is mysterious that Frege should introduce a universe of discourse yet never tell us that this is what he is doing. But the problems with this interpretation are not merely exegetical. The more serious issue is that, on this interpretation, the logic introduced in *Basic Laws* can not be used for the purpose for which it was intended. Frege's aim is to show that the truths of arithmetic analytic; that they can be given gapless proofs using only definitions and laws that govern all thought—that is, laws that apply to the objects of the empirical sciences, to the objects of geometry, to everything thinkable. If Frege has only shown that his Begriffsschrift laws apply to the objects of a limited universe of discourse, then derivations of truths of arithmetic from these laws will not show that these truths are analytic.

There is sense, as well, in which the logic of *Basic Laws* does not, on this view, do what it is supposed to do. The Begriffsschrift expression of each of his propositions, Frege says in *Basic Laws*, explicitly expresses "all of the conditions necessary to its validity"; there can be no "tacit attachment of presuppositions in thought" that the (BLA vol. I, p. vi). Yet, if the laws of *Basic Laws* are to be regarded as holding only for courses-of-values, this is a tacit presupposition. And it is a tacit presupposition that could have been made explicit. To do this, we need only replace Frege's statement of the law (' $a \rightarrow (b \rightarrow a)$ ') with

$$a \rightarrow (b \rightarrow a) \text{ or } \neg(\exists\phi)(a = \varepsilon\phi(\varepsilon)) \text{ or } \neg(\exists\phi)(b = \varepsilon\phi(\varepsilon)).$$

The assumption that it is appropriate to talk of domains in the metalanguage, but not to eliminate tacit presupposition from Begriffsschrift makes sense on the contemporary view of logical notation as a purely formal notation that is interpreted in natural language.

But this is an unFregean view of logical notation. For Frege, Begriffsschrift is a real language. Its expressive powers come into their own when the purposes are scientific and the aim is the expression of gapless proof. To make use of a metatheoretic background language that states presuppositions that remain unexpressed in Begriffsschrift, is to give up on one of the essential purposes of Begriffsschrift. It is not plausible to suppose that Frege made such a momentous change without acknowledging it explicitly. It is especially implausible in light of Frege's statements about definitions in §§56-67 of the second volume of *Basic Laws*. In these sections Frege criticizes mathematicians for defining functions only over limited domains. The addition function, he says there, must be defined even for the Moon as argument. Moreover, he objects to the strategy of 'piecemeal definition' – beginning with a definition of a term over a limited domain and then later defining it over a wider domain. Were Frege's intent, in the first volume of *Basic Laws*, to show only that his logical functions are defined for courses-of-values as arguments, this work would be subject to both criticisms. It is, of course, possible that Frege simply changed his view. Yet there is no evidence of this. The later discussions contain no acknowledgement of any such change. They are, moreover, entirely in line with the (earlier) view of Begriffsschrift as a means for eliminating tacit presupposition.

What, then, are we to make of Frege's offending statement: that if " $\Phi(\xi)$ " has Bedeutung and if every proper name that results from filling in the argument place with a bedeutungsvoll proper name names the True, then " $(a)\Phi(a)$ " names the True? The question is too difficult to be addressed adequately here. It will be worthwhile, however, to make a few brief comments. Frege does say several times (in §§8, 17 and 20) what, it seems, he ought: that " $(a)\Phi(a)$ " names the True provided the value of the function $\Phi(\xi)$ is the True for every *argument*. Since this is correct, and he is willing to say it, why should he have made the offending statement at all? There is no explicit answer in Frege's writings, nor does Frege acknowledge the difference between the two sorts of statements. However, it is not difficult to see at least one reason for the appearance of the offending statement. It is forced on Frege by the employment, in §31, of the §29 criteria for having Bedeutung. The statement is also useful for drawing attention to the relation that holds between Begriffsschrift expressions when one expression is a universal generalization and the other is an instance of the first. Moreover, it is also not obvious that the offending statement is as bad as it may seem at first. Frege does, after all, envision adding names to Begriffsschrift for the purposes of evaluating inferences in other sciences. Provided we can identify an object, we can stipulate that a proper name is to designate this object. Frege may, then, be considering also the bedeutungsvollen

proper names that can, in principle, be added to Begriffsschrift. Thus, unless Frege thinks there are objects that are, in principle, unnameable, his description is more or less correct.³⁸

VIII Conclusion

In this paper I have tried to show that some of the apparent difficulties with §31 can be addressed by taking Frege's remarks about the nature of his project more seriously than they are usually taken. In particular, I have tried to show that many of the apparent difficulties with §31 are consequences of the metatheory-reading. I would like to end with a few remarks about this reading.

Frege's task, as he characterizes it in *Foundations*, is to prove the truths of arithmetic from primitive truths. He writes:

[T]he aim of proof is, in fact, not merely to place the truth of a proposition beyond all doubt, but also to afford us insight into the dependence of truths upon one another. After we have convinced ourselves a boulder is immovable, by trying unsuccessfully to move it, there remains the further question, what is it that supports it so securely? The further we pursue these enquiries, the fewer become the primitive truths to which we reduce everything; (FA, §2).

What are the primitive truths to which Frege means to reduce the truths of arithmetic? I have argued that these truths are the basic logical laws. Frege states these laws in Begriffsschrift and, using these laws and his definitions, derives some of the primitive truths of arithmetic. On Frege's view, as long as we understand Begriffsschrift, we can see that the basic Begriffsschrift laws are true. And, as he also indicates, we can see this without relying on evidence of the senses or intuition. We thus have only to work through Frege's proofs to see that the truths of arithmetic are analytic.

What would be gained by introducing a metatheory? Some have argued that a metatheory would enable Frege to show that his logical language does what it is supposed to do—that the language will not leave us with expressions that do not designate anything. But, as we have seen, from Frege's point of view, this is something that cannot be proved. For Begriffsschrift satisfies Frege standards only if all its terms, including all its primitive terms (that is, all its terms for things that are logically simple) have Bedeutung. And any attempt to prove that a term for something logically simple has Bedeutung, whether it is a term of natural language or a term of Begriffsschrift, is doomed to failure. Natural language, Frege is the first to admit, has advantages that

³⁸ Heck (1997) adopts a view very close to this one.

Begriffsschrift does not. In a famous passage, he compares natural language to the eye—which is more useful for most of our purposes than a microscope, to which he compares Begriffsschrift. But to suppose that natural language is also superior for the purposes for which Begriffsschrift is formulated—to suppose that natural language is superior for purposes of explaining the primitive logical notions—is to give up on Frege's logical enterprise. It is not that it is impossible, given Frege's views, for there to be a science of semantics—although there are barriers—but that such a science is a special science not deserving of the appellation 'logic'. And it is not a special science to which Frege devotes himself in *Basic Laws*.

Bibliography

- Angelelli, I., ed. (1964): *Gottlob Frege, Begriffsschrift und Andere Aufsätze*, Hildesheim: Olms [BS]
- Dummett, M. (1978): "Frege's Distinction between Sense and Reference", in *Truth and Other Enigmas*,; Cambridge, Mass.: Harvard University Press, pp.116-144
- Frege, G. (1891): "Funktion und Begriff", *Jenaischen Gesellschaft für Medicin und Naturwissenschaft*, H. Pohle: Jena; trans. as "Function and Concept" in (McGuinness, 1984), pp. 1-31
- (1892a): "Über Sinn und Bedeutung", *Zeitschrift für Philosophie und philosophische Kritik* 100; trans. as "On Sense and Meaning" in (McGuinness, 1984), pp. 25-50
- (1892b): "Über Begriff und Gegenstand", *Vierteljahrsschrift für wissenschaftliche Philosophie* 16; trans. as "On Concept and Object" in (McGuinness, 1984), pp. 192-205
- (1893): *Grundgesetze der Arithmetik* Band I, H. Pohle, Jena; partially trans. in (Furth, 1964) [GGA vol. I]
- (1903a): *Grundgesetze der Arithmetik* Band II; partially trans. in (Geach & Black, 1980) [GGA vol. 2], pp. 137-58
- (1903): "Über die Grundlage der Geometrie", *Jahresbericht der Deutschen Mathematiker-Vereinigung* 12. Band; trans. as "On the Foundations of Geometry: First Series", in (McGuinness, 1984), pp. 319-324;368-375
- (1904): "Was ist eine Funktion?", *Festschrift für Ludwig Boltzmann*, J. A. Barth: Leipzig; trans. as "What is a Function?", in (McGuinness, 1984), pp. 656-666
- (1906): "Über die Grundlagen der Geometrie", *Jahresbericht der Deutschen Mathematiker-Vereinigung*, 15. Band; trans. as "On the Foundations of Geometry: Second Series", in (McGuinness, 1984), pp. 293-309; 377-403; 423-430

- (1923): "Gedankengefüge", *Beiträge zur Philosophie des deutschen Idealismus* 3. Band; trans. as "Compound Thoughts", in (McGuinness, 1984), pp. 36-51
- (1970): *Begriffsschrift, a Formula Language Modeled upon that of Arithmetic for Pure Thought*. In *Frege and Gödel: Two Fundamental Texts in Mathematical Logic*. Edited by Jean van Heijenoort. Cambridge, Mass.: Harvard University Press [BEG].
- (1980a): *The Foundations of Arithmetic*. Translated by J.L. Austin, 2nd rev. ed. Evanston, Ill.: Northwestern University Press [FA].
- Furth, M., ed. and trans. (1964): *Basic Laws of Arithmetic*, Berkeley and Los Angeles: University of California Press
- Gabriel, G. et al., eds., Kaal, H., trans. (1980): *Philosophical and Mathematical Correspondence*, Chicago: University of Chicago Press
- Geach, P. & Black, M., eds. (1980): *Translations from the Philosophical Writings of Gottlob Frege*, 3rd ed., Totowa, NJ: Rowman & Littlefield
- Heck, R. (1997): "Grundgesetze der Arithmetik I §§29-32", *Notre Dame Journal of Formal Logic* 38, pp. 437-74
- Hermes, H., et al., eds., Long, P. & White, R., trans. (1979): *Posthumous Writings*, Chicago: University of Chicago Press [PW]
- Hermes, H. et al., eds. (1983): *Nachgelassene Schriften*, Hamburg: Meiner Verlag [NS]
- McGuinness, B., ed. (1984): *Gottlob Frege: Collected Papers on Mathematics, Logic, and Philosophy*, Oxford: Blackwell
- van Heijenoort, J., ed. (1970): *Frege and Gödel*, Cambridge, Mass.: Harvard University Press [BS]
- Parsons, C. (1965): "Frege's Theory of Number", in C. Parsons, *Mathematics in Philosophy: Selected Essays*, Ithaca: Cornell University Press, pp. 150-75
- Resnik, M. (1986): "Frege's Proof of Referentiality", in J. Haaparanta & L. Hintikka, eds., *Frege Synthesized*, Dordrecht: Reidel, pp. 177-96
- Stanley, J. (1996): "Truth and Metatheory in Frege", *Pacific Philosophical Quarterly* 77, pp. 45-70
- Thiel, C. (1965): *Sinn und Bedeutung in der Logik Gottlob Freges*, A. Hain:Meisenheim am Glan; trans. as *Sense and Reference in Frege's Logic*, Dordrecht: Reidel, 1968
- (1975): "Zur Inkonsistenz der Fregeschen Mengenlehre", in C. Thiel, ed., *Frege und die moderne Grundlagenforschung*, Meisenheim am Glan: Hain, pp. 134-159
- (1996): "On the Structure of Frege's System of Logic", in M. Schirn, ed, *Frege: Importance and Legacy*, Berlin/New York: de Gruyter, pp. 261-279
- Weiner, J. (1990): *Frege in Perspective*, Ithaca, NY.: Cornell University Press
- (1995a): "Realism bei Frege: Reply to Burge," *Synthese*, March 1995, pp. 363-382

- (1995b): "Burge's Literal Interpretation of Frege", *Mind*, July 1995, pp. 585-597
- (1999): *Frege*, Oxford: Oxford University Press
- (forthcoming): "Understanding Frege's Project", in T. Ricketts, ed., *Cambridge Companion to Frege*, Cambridge University Press
- (in preparation), "Science and Semantics: the Case of Vagueness"