

Understanding Frege's Project¹

Frege begins *Foundations of Arithmetic*, the work that introduces the project which was to occupy him for most of his professional career, with the question, "What is the number 1?" It is a question to which even mathematicians, he says, have no satisfactory answer. And given this scandalous situation, he adds, there is small hope that we shall be able to say what number is. Frege intends to rectify the situation by providing definitions of the number one and the concept number. But what, exactly, is required of a definition? Surely it will not do to stipulate that the number one is Julius Caesar—that would change the subject. It seems reasonable to suppose that an acceptable definition must be a true statement containing a description that picks out the object to which the numeral "1" already refers. And, similarly, that an acceptable definition of the concept of number must contain a description that picks out precisely those objects that are numbers—those objects to which our numerals refer.

Yet, while Frege writes a great deal about what criteria his definitions must satisfy, the above criteria are not among those he mentions. Nor does he attempt to convince us that his definitions of "1" and the other numerals are correct by arguing that these definitions pick out objects to which these numerals have always referred. There is, as we shall see shortly, a great deal of evidence that Frege's definitions are not intended to pick out objects to which our numerals already refer. But if this is so, how can these definitions teach us anything about our science of arithmetic? And what criteria must these definitions satisfy? To answer these questions, we need to understand what it is that Frege thinks we need to learn about the science of arithmetic.

I

Why define the number one?

Definitions of the number one and the concept of number are necessary, Frege thinks, if we are to prove the truths of arithmetic from primitive truths. What are primitive truths? And why should we prove the truths of arithmetic from primitive truths? In the early sections of *Foundations*, Frege offers two motivations for attempting to provide such proofs. The first, which he characterizes as mathematical, is a desire for increased rigor—proof wherever proof is possible. The second, which he characterizes as philosophical, is a desire to show whether the truths of arithmetic are a priori or a posteriori, synthetic or analytic.

For Frege, the classification of a provable truth as analytic or synthetic, a priori or a posteriori, is determined by its most economical (most general) proof -- by the proof requiring the fewest specific assumptions. The least economical (least general) sort of proof is one that requires an appeal to facts, that is, unprovable truths about particular objects; unprovable truths that are not general. Appeals to facts are required by any proof of an a posteriori truth.² Truths of empirical science are examples of a posteriori truths. A truth that can be proved without appeal to facts is a priori and can be either synthetic or analytic. This classification, again, depends on what sort of proof is available. An a priori truth is synthetic if it cannot be proved "without making use of truths which are not of a general logical nature, but belong to the sphere of some special science".³ Truths of Euclidean geometry are examples of synthetic a priori truths. For the axioms from which they are derived are not of a general logical nature (they govern a limited domain: that of spatial configurations) but are general (they are not truths about particular objects). Finally, an analytic truth can be proved using only "general logical laws and

definitions". This is the most economical (or general) sort proof—it requires no appeals to facts or to truths of a special science.

To find the most economical proof of some truth, we need a method for recognizing gapless proofs. Otherwise, we cannot rule out the possibility that a proof that apparently has only general logical laws and definitions among its premises actually contains an implicit appeal to something that is neither a logical law nor a definition. The task is "that of finding the proof of the proposition, and of following it up right back to the primitive truths."⁴ In the process, Frege says,

we very soon come to propositions which cannot be proved so long as we do not succeed in analysing concepts which occur in them into simpler concepts or in reducing them to something of greater generality. Now here it is above all Number which has to be either defined or recognized as indefinable.⁵

One might suppose that, in this process, the concept of number *will* be recognized as indefinable. Yet Frege insists on defining the concept of number and the numerals. Why?

If the point of proving truths of arithmetic from primitive laws is to enable us to determine the correct classification of these truths, there will be eligibility conditions that determine what can be taken as a primitive law. One obvious eligibility condition is that its truth be evident without proof.⁶ Another is that there be some means, other than examining a proof, of determining its classification (i.e, of determining whether it is a fact about particular objects, a primitive general truth of some special science, or a general logical law). For if there is to be a definite answer to the question about the correct classification of the truths of arithmetic, then there must be some means of classifying the primitive laws on which the truths of arithmetic

depend.

To see how this might work, consider an example: the claim that every object is identical to itself.⁷ Since its truth is self-evident, it satisfies the first eligibility requirement for primitive laws.⁸ Supposing this to be a primitive law, is it analytic? In *Foundations*, Frege mentions two features of analytic truths. One is maximal generality. Analytic truths govern "not only the actual, not only the intuitable, but everything thinkable".⁹ Another is that we cannot deny them in conceptual thought. That is, we cannot deny them "without involving ourselves in any contradictions when we proceed to our deductions"¹⁰. Fundamental truths of arithmetic seem to be analytic because if we try to deny any one of them "complete confusion ensues. Even to think at all seems no longer possible."¹¹ The law that every object is identical to itself exemplifies both of these features. First, this law surely tells us, not just about every actual (spatio-temporal) object or every intuitable object, but about *every* object. Second, it seems that we cannot deny it without involving ourselves in contradictions. Given these criteria, the law in question is analytic. The axioms of geometry, in contrast, are synthetic because we *can* assume the contrary of an axiom of geometry without involving ourselves in contradictions.¹²

What of basic truths about numbers? Frege suggests, without argument, that the fundamental propositions of the science of number have the same status as logical laws -- that denying them will involve us in contradictions.¹³ He also states, again without argument, that the truths of arithmetic govern the widest domain of all [*das umfassendste*]. Thus these truths seem to be logical laws. But there are also reasons for thinking that they are not primitive logical laws. Truths about the number one do not seem to have the requisite maximal generality of logical truths. The number one, after all, is a particular object. Nor do laws about numbers

seem maximally general. They seem to govern, not the widest domain of all, but the peculiar domain of numbers. Inferences by mathematical induction appear to be "peculiar to mathematics."¹⁴ To substantiate his conviction that the truths of arithmetic are analytic, Frege needs to define the number one and the concept of number from recognizably logical notions and to prove the truths of arithmetic using only these definitions and logical laws.

II

Definitions and content

We now have one criterion that Frege's definitions must satisfy. They must enable him to provide gapless proofs of the truths of arithmetic from primitive truths—from primitive logical laws, if he is to show that they are analytic. Since the proofs must be of truths of *arithmetic*, Frege's definitions must not transform arithmetic into some new and foreign science. One might suppose, then, that Frege's aim is to give descriptions that pick out the objects that we are already talking about when we use the numerals and the term 'number'. Why, then, does he not say so?

The explanation, one might suspect, is simply that Frege expected this to be obvious to his readers. But a problem remains. If this is right, Frege's defense of his definitions should include an attempt to show that his definitions pick out the objects to which our numerals already refer. But Frege's *Foundations* defense of his definitions includes no argument that they pick out the objects to which our numerals already refer. What is Frege's defense?

The defense in *Foundations* appears in a group of sections labeled 'the completion and testing [*Ergänzung und Bewährung*] of our definition'. He first defines the concept *number* which belongs to the concept *F* and shows that this definition passes several tests. He then turns

to the task of completing his definitions—defining the individual numbers—which is followed with tests of these definitions. The tests are tests of whether the definitions allow us to derive "the well known properties of numbers".¹⁵ What are these properties? Although Frege is renowned for claiming that numbers are non-spatio-temporal objects, this is not the sort of property that must be derivable from the definitions. Rather, the properties in question are those that seem to underlie the uses we make of arithmetic, both in science and in everyday life. For example, we must be able to prove, using his definitions, that 0 is the number belonging to a concept if and only if no object falls under it (the number that belongs to a particular concept is the number of objects that fall under the concept); that if 1 is the number which belongs to a concept, then there exists an object which falls under that concept..¹⁶ The definitions must provide a basis for an arithmetic that meets the demand "that its numbers should be adapted for use in every application made of number".¹⁷ Thus the definitions must be responsible to pure arithmetic. For the applications of arithmetic are applications of pure arithmetic. If Venus has 0 moons and the Earth has 1 moon, we ought to be able to infer that the Earth has more moons than Venus—something that would be blocked were it a truth of Fregean arithmetic that $0=1$. What we take to be simple truths and applications of our arithmetic must be reproducible in an arithmetic based on Frege's definitions. No acceptable definitions of '0' and '1' will make it true that $0 = 1$ or false (failing new astronomical events) that the Earth has 1 moon. Moreover, it will not suffice that it be *true*, given Frege's definition of '1', that if 1 is the number which belongs to a concept, then some object falls under the concept. It must be *derivable*. The definitions must not only preserve what we regard as the truths of pre-systematic arithmetic, they must also provide support for its inferences. That is, the introduction of these definitions should enable us

to replace our original, enthymematic arguments about, say, the numbers of moons of Venus and the Earth, with gapless arguments.¹⁸

Definitions satisfying these constraints clearly preserve some pre-systematic content associated with the numerals and the term 'number'. This content seems very like the kind of content Frege introduces in *Begriffsschrift*: conceptual content [*begriffliche Inhalt*], content that has “significance for the inferential sequence”. And *Begriffsschrift*, the language in which he wants to carry out the proofs that will establish the analyticity of arithmetic, is designed to be a language that expresses conceptual content. This suggests that the criterion that must be met, if we are legitimately to regard Frege's definitions as faithful to arithmetic, is that they preserve whatever conceptual content is inherent in our pre-systematic views about arithmetic.

Although Frege does not use the expression 'conceptual content' in *Foundations*, the link between the content he wants to preserve and significance for inference is evident. He wants to convince us that intellectual effort is needed if we are to understand the content of the expression 'number' and the numerals. And one sort of evidence he offers is that, while we routinely make everyday inferences from numerical formulae, these inferences do not seem immediately licensed by logical laws. Nor is it evident how these inferences can be made gapless. The link between content and inference is also apparent in other discussions of *Foundations*. For example, in his discussion of the content of the proposition "All whales are mammals", he argues that the proposition is not about animals because,

We cannot infer from it that the animal before us is a mammal without the additional premiss that it is a whale, as to which our proposition says nothing.¹⁹

He also argues that the ideas we associate with an expression cannot constitute its content,

because the associated ideas do not support our inferences. Thus Frege's explicit requirements on his definitions involve preservation of whatever conceptual content is already associated with the word 'number' and the numerals.

III

Definitions and reference

But we might well have expected Frege to require that definitions of the numerals pick out whatever it is that we have been talking about all along. Or, to use a contemporary locution: that definitions of the numerals preserve pre-systematic reference. Yet Frege not only fails to articulate this requirement, he makes no attempt to show that his definitions satisfy it. One might suspect that he simply assumes that, to show that the definition picks out the object we have been talking about in our use of the term '1', it will suffice to show that the definition preserves the conceptual content of '1'. But Frege's actual remarks suggest something very different: that the terms to be defined do not actually have reference antecedent to his work.

Frege writes, in a criticism of a proposed definition of the concept of number, "it must be noted that for us the concept of number has not yet been fixed, but is only due to be determined in the light of our definition of numerical identity".²⁰ This is an odd choice of words if each numeral *already* refers to a particular object and if to be a natural number is simply to be one of these objects. For Frege writes that all that can be demanded of a concept is that it should be determined, for each object, whether or not it falls under the concept.²¹ If each numeral *already* refers to a particular object, and if the numbers are the objects to which the numerals refer, then the concept of number is already fixed. We may be lacking a definition that identifies this fixed concept of number. But it certainly does not follow that the concept of number is due to be

determined in the light of our definition. Were this Frege's only remark of the sort, one might dismiss it as merely an odd choice of words. But it is not.

Most of the discussions of *Foundations* are about the natural numbers. However, Frege also discusses the complex numbers. He considers the possibility of stipulating that the time-interval of one second is the square root of -1 , and adds, in a footnote, that we are entitled to choose any one of a number of objects to be the square root of -1 . The reason is that

the meaning [*Bedeutung*] of the square root of -1 is not something which was already unalterably fixed before we made these choices, but is decided for the first time by and along with them.²²

If this is so, our symbols for complex numbers do not already refer to particular objects—which he goes on to suggest in the next section. In this remark, unlike the earlier remark about the concept of number, there is no ambiguity. One might suspect that this marks a difference between the complex and natural numbers. But Frege gives us no indication that there is such a difference.

He goes on to suggest that there is a problem with defining the square root of -1 as a time-interval. This would import into arithmetic “something quite foreign to it, namely time” and make arithmetic synthetic.²³ To show that arithmetic is analytic, Frege proposes using the same solution for complex numbers that he used for natural numbers: to define them as extensions of concepts. The notion of extension of concept is a logical notion, on Frege's view, and definitions of numbers as extensions of concepts should make it possible to prove truths about numbers from logical laws. He ends *Foundations* with the following remark about offering such definitions:

Once suppose this everywhere accomplished, then numbers of every kind, whether negative, fractional, irrational or complex, are revealed as no more mysterious than the positive whole numbers, which in turn are no more real or more actual or more palpable than they.²⁴

This would be an odd remark if, for example, '1' had *all along* referred to a particular extension of a concept while the symbol 'i' refers to an extension of a concept only because of an arbitrary stipulation. But, again, this may be simply an odd choice of words. What other evidence is available?

Frege acknowledges that the correctness of his definitions is not evident. For we "think of the extensions of concepts as something quite different from numbers".²⁵ One might expect him to go on to argue that numbers really are extensions of concepts. But he does not. Rather, he claims that he attaches no decisive importance to bringing in the extensions of concepts.²⁶ This is completely mysterious if we assume that, when we use the numerals in our current pre-systematic language, we are talking about particular objects, and if we assume that Frege's task is to provide definitions that pick these objects out. Given these assumptions, either we are already talking about (our numerals already refer to) extensions of concepts (in which case it would be *essential* to bring in extensions) or we are already talking about (our numerals already refer to) objects other than extensions of concepts (in which case it would be *wrong* to bring in extensions). Frege's comments are simply not consistent with the assumption that his definitions are meant to pick out objects that we have been talking about all along. Unless we are prepared to engage in interpretive contortions, the appropriate conclusion is that, when Frege asks for a definition of the concept number, he is not asking for explicit descriptions of objects to which

our numerals already refer. And, given this, it is implausible to attribute to Frege the view that there is a concept to which 'number' refers, and objects to which the numerals refer, antecedent to his introduction of his definitions. That is, antecedent to Frege's introduction of his definitions, the concept number is not fixed.²⁷

Of course, if Frege's explicit remarks are absurd, there may be a compelling reason to engage in interpretive contortions. But are they? There are many distinct set theoretic definitions of the numbers that fit our understanding—both everyday and scientific—of the numbers. Nothing in our understanding of the truths of arithmetic seems to offer grounds for deciding between alternative systems of set theoretic definitions or, for that matter, grounds for saying that numbers are (or are not) sets. Given this, Frege's explicit remarks do not seem absurd at all. There is every reason to believe that the numerals do not refer to particular objects and, consequently, that the content associated with the numerals can be captured by offering definitions that are at least partly stipulative.

IV

Reference and truth

There is a problem, however. Frege seems to assume, not just that such everyday sentences of arithmetic as '0 is not equal to 1' express truths but that they express truths about particular numbers. Otherwise, what would be the point of defining the numbers as objects? But now suppose '0 is not equal to 1' expresses a true claim about particular numbers. It seems that its truth must depend on the character of those numbers—i.e., the character of the objects to which '0' and '1' refer. If there are no objects to which '0' and '1' refer, it follows that '0 is not equal to 1' does not express a truth. It seems to follow that no statements of everyday arithmetic can

express truths.

Frege never addresses this problem. The explanation, one might suspect, is that he simply did not notice this consequence of his views.²⁸ But this is not entirely convincing. For, he comes very close to explicitly acknowledging this consequence. He writes “would the sentence ‘any square root of 9 is odd’ have a comprehensible sense at all if *square root of 9* were not a concept with a sharp boundary?”²⁹ A sentence without comprehensible sense cannot have a truth-value. One might suspect that Frege takes it to be obvious that ‘any square root of 9 is odd’ does have a comprehensible sense and, hence, that the concept *square root of 9* does have a sharp boundary. However, a look at the context in which the question appears shows that this interpretation is incorrect. A concept has a sharp boundary just in case it determinately holds or not of each object. For example, in order for *greater than zero* (or *positive*) to be a proper concept, Frege says, “it would have to be determinate whether, e.g., the Moon is greater than zero”.³⁰ He continues,

We may indeed specify that only numbers can stand in our relation, and infer from this that the Moon, not being a number, is also not greater than zero. But with that there would have to go a complete definition of the word ‘number’, and that is just what is most lacking.

In the discussions of *Basic Laws* that immediately follow, he suggests that such expressions as ‘greater than’ and ‘+’ are used by mathematicians in such a way that they have no fixed meaning.³¹

It is difficult to imagine that Frege said all this without noticing the consequence that sentences in which ‘greater than 0’, ‘greater than’ and ‘+’ appear have no truth-value.³² Indeed,

given his requirement that each predicate pick out a concept with a sharp boundary, few, if any, of our everyday sentences have comprehensible sense or truth-values. But, whether he noticed this or not, this creates a puzzle about Frege's conception of his project. Frege's avowed project is to show that the truths of arithmetic are analytic. Unless we already know some of these truths—unless our everyday sentences of arithmetic express them—what could be the point of this project? Although Frege does not address this problem explicitly, there are solutions to it to be found in his discussions of natural language, Begriffsschrift, and science.

Frege characterizes Begriffsschrift, his logical language, as a tool that enables us to avoid some difficulties inherent in natural language. When we use natural language, he says, even careful use of logical laws will not prevent errors. Mistakes, he writes, "easily escape the eye of the examiner, especially those which arise from subtle differences in the meanings of a word".³³ He continues,

That we nevertheless find our way about reasonably well in life as well as in science we owe to the manifold ways of checking that we have at our disposal. Experience and space perception protect us from many errors.

Frege does not suggest that there is anything wrong with relying on the manifold ways of checking or that the subtle differences in the meanings of a word should be eliminated from natural language. Rather, these features of natural language are rooted "in a certain softness and instability of language which nevertheless is necessary for its versatility and potential for development".

In this respect, language can be compared to the hand, which despite its adaptability to the most diverse tasks is still inadequate. We build for ourselves artificial hands, tools

for particular purposes, which work with more accuracy than the hand can provide. And how is this accuracy possible? Through the very stiffness and inflexibility of parts the lack of which makes the hands so dextrous. Word-language is inadequate in a similar way. We need a system of symbols from which every ambiguity is banned, which has a strict logical form from which the content cannot escape.

Neither natural language nor a logically perfect symbolic language is suitable for every purpose.

Whether features of a language count as virtues or defects will depend on the purpose for which we want to use the language. Features of natural language that are defects, given Frege's specialized purposes, are desirable for other purposes. Begriffsschrift is not an ideal language. It is "a device invented for certain scientific purposes and one must not condemn it because it is not suited to others."³⁴

Begriffsschrift is designed for the expression and evaluation of inferences. It must be capable of expressing all content of any statement that has significance for the inferences in which it can figure. Once an inference is expressed in Begriffsschrift, the employment of Frege's logical laws and rules are to make it a mechanical task to determine whether it is correct and gapless, or whether it requires an unstated premise. Because the task is mechanical, no presupposition can sneak in unnoticed. We need such a language and logical system in order to produce identifiably gapless proofs of the truths of arithmetic. And only identifiably gapless proofs from primitive truths will enable us to determine whether the truths of arithmetic are correctly classified as analytic or synthetic.

In order to carry out this project, we must define all terms of arithmetic from primitive, undefinable terms and construct a list of axioms or primitive truths from which all truths of

arithmetic can be proved by gapless logical inferences. To do this is (to use Frege's later expression) to provide a systematic science. And science, Frege claims, comes to fruition only in a system.³⁵ Arithmetic is a science in its early stages—a science whose sentences have not yet been associated with precise thought content. It is not that that arithmetic is less developed than other sciences. Although it is as highly developed as any science, arithmetic does not satisfy the standards for systematic science. In fact, there are no systematic sciences—Euclidean geometry comes closest, but its proofs are not gapless.³⁶ Frege's systematic science of logic, of which arithmetic is a part, will be the first.

V

Natural language and Begriffsschrift

We can now see why it is not absurd for Frege to say that the everyday sentences of natural language do not have truth-values. Frege's view seems to be that truth is what we get, not in everyday circumstances, but rather at some ideal end of inquiry. And the language for this ideal end, the language for systematic science, is not natural language but Begriffsschrift. But while natural language may not be a good vehicle for expressing truth, it is an essential tool in the early stages of our attempts to express truths. In a diary entry, Frege wrote, of his attempt to say what the numbers are,

[O]ne might think that language would first have to be freed from all logical imperfections before it was employed in such investigations. But of course the work necessary to do this can itself only be done by using this tool, for all its imperfections. Fortunately as a result of our logical work we have acquired a yardstick by which we are apprised of these defects. Such a yardstick is at work even in language, obstructed

though it may be by the many illogical features that are also at work in language.³⁷

To systematize a science we begin with the everyday sentences that are regarded as its basic truths—such sentences as '0 is not equal to 1'. Our everyday view that this sentence expresses a truth is not quite right. The content associated with it is not yet precise enough; the science is not yet sufficiently well worked out. But much of the science of arithmetic is worked out. Many of the standards by which we judge sciences have been met. This sentence provides a guide for systematizing arithmetic. For it places constraints on our assignments of meaning to the terms '0' and '1'. On any acceptable assignment the sentence '0 is not equal to 1' must express a truth. Since it will help to have a label for this attitude in the discussion that follows, I will say that Frege *regards these sentences as true*. It is a consequence of Frege's view that few, if any, of our everyday sentences actually express truths. Nonetheless it is consistent with his view that we can regard some of these sentences—particularly the results of pre-systematic research—as expressing truths.

One might suspect that this view must conflict with Frege's statements in "On Sense and Bedeutung", which includes extensive discussion of natural language. Frege introduces his renowned Sinn/Bedeutung distinction by talking about words and sentences of everyday language. And the Bedeutung of an object expression is whatever object that expression designates. Yet it is difficult to find any actual inconsistency. Although Frege writes as if the terms of everyday language have Bedeutung and the sentences of everyday language have truth-values, he never actually *says* that they do. It is not because the subject never comes up. Although he raises the question of whether "the Moon" has a Bedeutung, he does not go on to say that it does. He says only that we "presuppose a meaning [*Bedeutung*]".³⁸ Nor does he say

that such presuppositions are always—or generally, or even *sometimes*—correct. He says only, the question whether the presupposition is perhaps always mistaken need not be answered here; in order to justify mention of that which a sign means it is enough, at first, to point our intention in speaking or thinking. (We must then add the reservation: provided such a meaning exists.)³⁹

We do, of course, presuppose that our terms have *Bedeutung*, that there is something we really are talking about and that our sentences really have truth-values.⁴⁰ As we have seen, Frege's comments about the numerals and 'number' indicate that he thinks there are scientific contexts in which this presupposition is incorrect. The incorrectness of this particular presupposition has not, however, impeded our everyday arithmetic. It has not even impeded such sophisticated mathematical uses as Weierstrass'.⁴¹ It is the project of systematization that requires both that all presuppositions be eliminated and that the necessary work be done to guarantee that each term has *Bedeutung*.

This is not to say that *Bedeutung* is unimportant in our use of natural language. But it is not a prerequisite for our use of natural language—even in scientific contexts—that our terms have *Bedeutung*. But is this view plausible? Surely, one might think, it is essential that terms used in scientific contexts have *Bedeutung*. In fact, however, this apparently implausible view, at least in some cases, fits our conception of good scientific practice perfectly well. To see this, it will help to look at an example.

Today, as a result of a good deal of research, it is widely regarded as a well-established truth that obesity increases one's risk of heart disease. Yet 'obese' no more designates a fixed concept than 'number'. Although medical researchers studying obesity agree that obesity is some

weight-related characteristic that is associated with increased morbidity and mortality, several distinct sorts of definitions are used in medical research. Most common, because of convenience, are definitions in terms of body mass index, an index calculated using measurements of height and weight.⁴² The current general acceptance of definition of obesity as BMI>30—by researchers, the World Health Organization, public health officials, newspaper reporters and their readers— is a fairly recent phenomenon. Only 15 years ago the preferred definition was a two-part definition: for men the obesity began at BMI>28.7, for women BMI>28.3.⁴³ And it is also widely acknowledged that the current definition is not ideal. For almost everybody believes that obesity has something to do with body fat and some highly muscled athletes who do not have much body fat will be classified as obese, given this cut-off.⁴⁴

The search for a good definition of obesity continues, along with the investigation of various hypotheses about obesity. Yet it would be unreasonable to halt all investigation of the effects of obesity on morbidity and mortality on the grounds that, since the concept has yet to be fixed, the hypotheses have no truth-values. It would be unreasonable to give up our view that it is *true* that obesity increases risk of heart disease. That is, an apparently absurd view that Frege seems to hold—that we are entitled to regard certain sentences as expressing truths, in spite of the fact that some of their terms do not have fixed meaning—is not absurd at all. It aptly describes perfectly unexceptionable views of researchers. But this is not to say that the issue of a term's having fixed meaning is of no concern to this sort of science. In fact, the problem with requiring that all terms used in scientific investigation already have fixed meaning is precisely that it can be part of the scientific enterprise to fix the meaning associated with a term already in

use. The procedure, as we have noted already, involves a combination of research and stipulation.

What, then, is Frege's view of truth? He may seem to have two notions: the strict sort of truth that is the aim of science and a different sort of truth that applies to sentences of natural language—something very like the supervenientist notion of truth.⁴⁵ After all, the significance of his regarding it as true that each number has unique successor is that on every acceptable definition of the term 'number', it will be provable, hence true, that each number has a unique successor. There is, however, an important difference between Frege's and the supervenientist's views of natural language. Although Frege shares the supervenientist view that there is something right about many of our everyday sentences, he does not share the supervenientist we are correct to presuppose that the constituents of these sentences have fixed meaning. For while there is something right about the sentences that we regard as setting out fundamental truths of pre-systematic arithmetic, the demands of truth, as Frege understands them, show us that there is also something wrong with these sentences. Frege wants to satisfy these demands, using what is right about pre-systematic arithmetic as a starting point.

Frege wants to replace imprecise pre-systematic sentences with precise systematic sentences—e.g. to introduce definitions of 'number' and 'successor', from which it can be proved that each number has unique successor. For Frege's interest in "the sort of truth which it is the aim of science to discern" will not allow him to rest content with the standards of pre-systematic arithmetic.⁴⁶ To say that our statements do not now satisfy Frege's demand that all constituents have fixed meaning is merely to say that we are not finished. Our sciences have not yet reached fruition. The demands that Frege identifies as the demands of truth should be seen as part of a

regulative ideal for science. But there is no reason to assume that any sentences of natural language actually satisfy the demand. Thus we can reconcile Frege's conception of his project with his statements about truth. To show that the truths of arithmetic are analytic is not to undertake a project external to the development of the science of arithmetic. It is a further—the final—step in bringing this science to fruition.

VI

Semantic Descent

There is, for Frege, only one notion of truth. It is what we obtain in systematic science. And Frege does not think this is an unobtainable ideal. His new logic is meant to be a systematic science. Moreover logic, he tells us, has a special relation to truth: the task of logic is to discern the laws of truth. Thus Frege's logic seems to give us some sort of theory of truth. But what is this theory like? It is widely supposed that Frege means to give a theory that tells us how the truth of a sentence is determined by semantic values of its subsentential constituents. Of course, no such theory is stated in *Begriffsschrift*. But the *Begriffsschrift* proofs, many think, are only one part of Frege's logic. His logic, on this view, is a familiar enterprise that involves a formal language and its interpretation; it is a science in which metatheory and metatheoretic proof play important roles. There are, however, a number of difficulties with this reading.⁴⁷

One difficulty lies in the significance accorded to language. If a theory of truth tells us how the truth of a sentence is determined by semantic values of its subsentential constituents, then language would appear to be the subject of the theory of truth. Moreover, language appears to be the subject of most metatheoretic proof. Consider, for example, the sort of metatheoretic justification one might offer for Modus Ponens:

if A is true and $A \rightarrow B$ is true, then B is true.

This statement appears to make a general claim about sentences, with 'A' and 'B' used as metatheoretic variables that range over sentences. But as we saw earlier, Frege believes that laws of logic are distinguished by their universality. They hold, not just over the realm of some special science or the realm of the spatio-temporal, but over an unrestricted realm. How can the metatheoretic claim about Modus Ponens—which appears to be, not a statement about everything thinkable, but a statement about the restricted realm of the linguistic—justify truths that hold over an unrestricted realm?

Quine, who also claims that language is not the subject of logic, offers us a familiar answer.⁴⁸ He points out that “‘Wombat’ is true of some creatures in Tasmania”, which is about a linguistic expression, is also a paraphrase of “There are wombats in Tasmania”, which is not. Thus it is possible to use statements about language to express something whose subject is really not language at all. The strategy of talking about words when our actual interest is in something else, which Quine labels *semantic ascent*, is, he argues, necessary for logic. Logic “can be expounded in a general way *only* by talking of forms of sentences”.⁴⁹

The reason stems from the sorts of generalizations required by logic. Consider the clause 'time flies' in the sentence 'if time flies then time flies'. Quine writes,

We want to say that this compound sentence continues true when the clause is supplanted by any other; and we can do no better than to say just that in so many words, including the word 'true'. We say "All sentences of the form 'if p then p' are true." We could not generalize as in 'All men are mortal', because 'time flies' is not, like 'Socrates', a name of one of a range of objects (man) over which to generalize. We cleared this obstacle by

semantic ascent by ascending to a level where there were indeed objects over which to generalize, namely linguistic objects, sentences.⁵⁰

On Quine's account, semantic ascent solves a problem. Semantics is required for logic because the generalization needed for a general account of the logical laws is not generalization over objects.

To see how this works, consider a contemporary rendering of Frege's Basic Law 1, ' $(A \rightarrow (B \rightarrow A))$ '. The contemporary rendering, is neither an expression in contemporary logical notation nor a single logical law. It is, rather, a schema in which 'A' and 'B' are used as metalinguistic variables that range over sentences. A claim about the truth of Basic Law 1 is really a claim about the truth of infinitely many sentences in the formal language. Similarly, the statement that explains the justification of Modus Ponens is the statement of a general claim about sentences and truth: for any sentences A and B, if A is true and $A \rightarrow B$ is true, then B is true. One hallmark of contemporary logic, then, is the use of schemata. Michael Dummett writes, "Logic can begin only when the idea is introduced of a schematic representation of a form of argument".⁵¹

Another hallmark of contemporary logic is the use of the truth predicate, where truth is a property of sentences.⁵² Quine writes that the truth predicate has its utility, in just those places where, though still concerned with reality, we are impelled by certain technical complications to mention sentences. Here the truth predicate serves, as it were, to point through the sentence to the reality; it serves as a reminder that though sentences are mentioned, reality is still the whole point.⁵³

It makes sense, on this view, to talk of the laws of logic as the laws of truth and it makes sense to

think that any general account of the logical laws must be metatheoretic. How close is this to Frege's view?

Some differences between the contemporary versions of the logical laws and rules and Frege's versions are purely notational, but others are not. To understand these differences and their significance, it will help to look at some of the discussions from the early sections of *Basic Laws*—the sections containing Frege's introduction and defense of the second version of his new logic.⁵⁴ As we have seen, a metatheoretic justification of Modus Ponens involves both the use of schemata to generalize over linguistic entities and a truth predicate. Frege's explanation, in its entirety, is:

for if Γ were not the True, then since Δ is the True [*das Wahre ist*] $\Delta \rightarrow \Gamma$ would be the False.⁵⁵

Does Frege use a truth predicate? The only candidate for a truth predicate in the above passage is the expression 'is the True' [*das Wahre ist*]. To see whether this is simply a peculiarly worded truth predicate, we need to look at Frege's use of the expressions 'the True' and 'the False'.⁵⁶

Frege introduces the True and the False in order to make out his claim that a concept is a sort of function.⁵⁷ But why take concepts to be functions? To define a function is to indicate what values it has for each argument. And a concept definition does not seem to give values for arguments but, rather, an indication of what falls under the concept. However, we might think of a concept as a function that gives us something for each object—either the answer 'true' or the answer 'false'. Taking this a bit further, Frege writes, of concepts,

I now say: 'the value of our function is a truth-value', and distinguish between the truth-values of what is true and what is false. I call the first, for short, the True; and the

second, the False.⁵⁸

Concept expressions are predicates. Thus the expression for the value a concept has for a particular object will be a sentence. For example, '2 is a prime number' is an expression for the value the concept *prime number* has for 2. Since 2 *is* a prime number, '2 is a prime number' designates the True, as do all other true sentences. Similarly, all false sentences designate the False. Frege's strategy for assimilating concepts to functions is to assimilate sentences to proper names.

As we saw earlier, the technique of semantic ascent is needed in contemporary logic because there is an obstacle: we cannot generalize over slots occupied by sentences because sentences are not proper names. But sentences *are* proper names for Frege. And, consequently, there is no such obstacle for Frege. An upshot is that Frege has no need for one of the essential elements of a metatheoretic soundness proof: a truth predicate. And there is no truth predicate (that is, no predicate that holds of true sentences) in Frege's discussion of Modus Ponens.⁵⁹ For 'is the True' is not a predicate that holds of all true sentences.

To see why, consider Frege's statements about sentences and proper names. In "On Concept and Object", Frege writes,

...a name of an object, a proper name, is quite incapable of being used as a grammatical predicate.⁶⁰

It is not, Frege continues, that we cannot use predicates in which a proper name follows 'is' (for example, 'is Venus'). It is that in these predicates, 'is' is not the copula but, rather, the 'is' of identity. Since the True is an object, 'the True' is an object name. It follows that the 'is' in 'is

the True' is the 'is' of identity. That is, the predicate 'is the True' means the same as 'is identical to the True'.

These views are repeated in *Basic Laws*. Frege introduces the truth-values (the True and the False) as objects.⁶¹ The view that the 'is' in the predicate 'is the True' is the 'is' of identity, comes out the use of the predicate 'is the True' to explain the horizontal. The horizontal is offered as a Begriffsschrift translation of 'is the True'. Frege writes,

— Δ

is the True [*das Wahre ist*] if Δ is the True; on the other hand it is the false if Δ is not the True [*nicht das Wahre ist*].⁶²

Moreover, the 'is' in 'is the True' must be the 'is' of identity. For he continues,

Accordingly,

— ξ

is a function whose value is always a truth-value—or by our stipulation, a concept.

Under this concept falls the True and only the True.⁶³

That is, 'is the True' is a predicate that holds of one object (the True) and no other.⁶⁴ Thus, since there are distinct true sentences, the predicate 'is the True' cannot hold of all true sentences.⁶⁵

And it is, as we have seen, 'is the True' rather than 'means (or denotes) the True' [*bedeutet das Wahre*] that appears in Frege's discussion of Modus Ponens.⁶⁶

What of the capital Greek letters that appear in Frege's defense of this rule? Are they used as metalinguistic variables? A moment's thought should show that they cannot be—since 'is the True' cannot hold of true sentences. But if the capital Greek letters that appear in Frege's discussion of Modus Ponens are not to be understood as generalizing over linguistic expressions,

how are they to be understood? The quick answer is that the generalization involved is no different from any generalization over objects. To see this, consider, again, Frege's introduction of the horizontal. To define a first-level concept, is to indicate, for each object, whether or not it holds of that object. By telling us that the horizontal names a concept that holds of the True and only the True, Frege does just that—both for objects named by sentences and for objects not named by sentences. He then goes on to say what the expressions '—2', '—2² = 4', and '—2² = 5' name. If Δ is not the True, — Δ is the False. Thus, given 2 is not the True, —2 is the False. Similarly, since the Moon is not the True, —(the Moon) is the False. That is, the capital Greek letters that appear in Frege's statements are not special metalinguistic variables. The generalization in the statements in which they appear is generalization over all objects.⁶⁷

Thus, Frege's discussion of why Modus Ponens is a good rule, unlike the kind of metatheoretical justification that appears in contemporary soundness proofs, exploits no truth predicate and no metalinguistic variables. But the rule itself differs only notationally from the contemporary rule. Frege's statements of his laws, in contrast, are different from contemporary laws. Because his actual symbols are difficult to print, I will continue using the contemporary arrow, rather than Frege's condition stroke, in the discussion of this rendering, but I will now add some of the requisite horizontals. Frege's assertion of Basic Law 1 looks something like this:

$$\vdash (—a \rightarrow (—b \rightarrow —a)).^{68}$$

But Frege's rendering, unlike the contemporary rendering, is not to be understood as a metatheoretic claim about infinitely many logical laws: a claim that “—(—a → (—b → —a))” *turns into* a true Begriffsschrift sentence whenever appropriate expressions are substituted for 'a' and 'b'. Frege's Basic Law 1 is a single law directly expressible in Begriffsschrift. The law is

simply a universal generalization "for any a and b...". A more revealing rendering of the content of Frege's law, using the peculiar notation I have introduced above, might be:

$$\vdash (a) (b) (\neg a \rightarrow (\neg b \rightarrow \neg a)).$$

Given this machinery, it should not be particularly surprising that no truth predicate appears in Frege's discussion of Basic Law 1.

Given Frege's assimilation of sentences to proper names, there is no need for semantic ascent in the discussion of the justification of the basic logical laws and rules.⁶⁹

VII

Why Avoid Using a Truth Predicate?

It would obviously be anachronistic to read Frege's work as offering a critique of the conception of logic on which metatheory plays a central role. But there are reasons to think that he was actively seeking a means to minimize use of the predicate 'is true' in the discussions of the justification of his logical laws and rules. Given the absence of a need for (or conception of) semantic ascent, if a truth predicate (which holds of sentences) plays a central role in an account of the justification of primitive logical laws, then logic would seem not to have the requisite generality. Its subject matter would appear to be, not everything thinkable, but only the limited domain of the linguistic. Of course, appearances can be misleading. After all, Frege thinks that the laws of arithmetic, which seem to express the peculiarities of a restricted domain, are logical laws. On the other hand, the fact that these laws seem to express peculiarities of a restricted domain is part of Frege's motivation for undertaking his project. One aim of Frege's proofs is to unmask the truths of arithmetic—to exhibit their true nature. Given the importance of this sort of unmasking, it should be no surprise that Frege would want to avoid justifications of the rules of

inference and basic laws of Begriffsschrift that seem to involve truths about the specific domain of the linguistic. The laws of truth should be laws that clearly do hold everywhere.

What, then, are we to make of Frege's statements that our conception of the laws of logic is connected with how one understands the word "true";⁷⁰ that the laws of logic are the laws of truth?⁷¹ Taken in isolation, these statements suggest that the laws of logic either are, or are justified by, general statements or laws in which the predicate 'is true' appears. However, there is no such suggestion in the context in which these statements appear. When Frege says in *Basic Laws* that the laws of logic are the laws of truth, he says this by way of warding off the interpretation of the laws of logic as psychological laws; the laws in accord with which we think. Thus he writes, "I understand by 'laws of logic' not psychological laws of takings-to-be-true, but laws of truth."⁷² They are "guiding principles for thought in the attainment of truth"⁷³. But this does not distinguish laws of logic from laws of other sciences. The laws of geometry and physics, Frege says, are also laws of thought in this sense.⁷⁴ They differ from laws of logic only in applying over more limited domain. The laws of geometry, for example, are guiding principles for thought in the attainment of truth about the peculiarities of what is spatial. The laws of logic, in contrast, are guiding principles for thought in all domains, they are laws that hold everywhere: or, simply, the laws of truth.⁷⁵

But how, if not via metatheoretic proof, can he convince us that these laws are both true and universal? Consider, again, the law that every object is identical to itself. Frege says it is impossible for us to reject this law. And it is evident that this law holds, not just over the limited domain of some special science, but over everything. If so, to see that a Begriffsschrift proposition expresses this law is to see that it expresses a logical law. The same should hold for

any primitive logical law we can identify. Similarly, one might suppose that anyone who understands Begriffsschrift will simply recognize the Begriffsschrift rules as correct rules of logic. Frege seems to have thought so when he wrote *Begriffsschrift*. For all he says there in defense of Modus Ponens is that its correctness is apparent from his explanation of the condition stroke. A lucid introduction of Begriffsschrift should suffice to convince the reader that the primitive Begriffsschrift laws and rules are logical laws and rules.

This is not to say that Frege has no metatheoretic perspective in any sense at all. It would be ridiculous to suggest that there is any way to introduce an artificial language such as Begriffsschrift without using everyday natural language to assign meanings to its symbols. Insofar as natural language discussion of Begriffsschrift belongs to metatheory, there can be no question that Frege's logic involves metatheory. Moreover, Frege certainly makes some arguments in natural language about the characteristics of his formal system.⁷⁶ These are metatheoretic arguments. But these arguments are no part of a foundation for logic—the foundation is simply the primitive logical laws and rules.

But, if this is so, what is the purpose of the (often elaborate) discussions of the truth of his basic laws in the early sections of *Basic Laws*? Why would he not simply introduce the primitive terms, list the axioms and rules and get immediately to work on the Begriffsschrift proofs?

VIII

Theory and Elucidation

Frege frequently remarks that the meaning of primitive terms can only be communicated via hints or elucidations. For example,

Since definitions are not possible for primitive elements, something else must enter in. I call it elucidation. It is this, therefore, that serves the purpose of mutual understanding among investigators, as well as of the communication of the science to others. We may relegate it to a propaedeutic. It has no place in the system of a science; in the latter, no conclusions are based on it.⁷⁷

But it is not much help simply to provide a label. What is elucidation?

Definitions are statements in a theory that are designed to communicate the meanings of terms. There are rules for a properly formed definition, rules designed to guarantee that the definition fixes the meaning of a term. Thus it is tempting to suppose that there will also be rules designed to guarantee (or at least make probable) the success of elucidation—albeit different and, perhaps, less reliable rules. But Frege also describes some elucidations as hints. After saying that we must rely on elucidation in our introduction of primitive terms, he also says “we must be able to count on a little goodwill and cooperative understanding, even guessing”⁷⁸ There are, then, no rules for successful elucidation.

Not is elucidation a technique for effective communication of the meaning of primitive terms. There is no such technique—as Frege makes clear in some of his discussions of logically simple notions. An example is his discussion of the notions of concept and object. Frege appears unperturbed by his recognition of the apparently paradoxical nature of his remarks and goes on, notoriously, to claim that some of his statements must either be false or miss his thought and to ask his readers to grant him a grain of salt. This would be mysterious were we to interpret Frege as giving an argument designed to establish, as its conclusion, one of his general remarks about the nature of concepts. But these remarks are, rather, part of Frege's elucidatory attempt to

communicate an understanding of the notions of concept and object. If we can accept Frege's characterization of elucidations as hints, his odd attitude is explicable.⁷⁹

This is not to say that elucidation *must* consist of apparently paradoxical utterances or failed attempts to express the inexpressible. Indeed, most of Frege's elucidatory remarks are entirely unproblematic. There is nothing paradoxical about Frege's claim that the singular definite article indicates an object. Nor is this a failed attempt to express the inexpressible. But this claim, like the apparently paradoxical claims about what it is to be concept, is no part of Frege's systematic science. There is no Begriffsschrift expression for predicating objecthood, hence no logical law that tells us what it is to be an object. Frege's introduction of the term 'elucidation' is meant to highlight the difference between these attempts to communicate the meanings of terms and actual definitions. Statements that appear in discussions belonging to the propaedeutic of a theory are to be distinguished from actual propositions of the theory.

What does this understanding of the role of elucidation in Frege's project tell us about his discussions of the justification of his primitive laws and rules? Consider, first, the laws. Frege writes,

The questions why and with what right we acknowledge a law of logic to be true, logic can answer only by reducing it to another law of logic. Where that is not possible, logic can give no answer.⁸⁰

As we have seen, Frege indicates that we cannot doubt primitive logical laws; that all we need in order to see that the primitive laws on which he relies are true is an understanding of the Begriffsschrift terms used in their statement. He also claims, later in his career, that the denial of a logical law can appear, if not nonsensical, at least absurd.⁸¹ In this case logic, it would seem,

can (and need) give no answer. All we need, it seems, is elucidation.

But can Frege's actual discussions be elucidations? After all, they do not introduce primitive terms but, rather, are attempts to show that the complex expressions he uses to express his primitive laws express truths. Moreover, they seem to have the character of arguments, not hints.⁸² A closer look at Frege's writings shows us that he neither restricts elucidation to the introduction of primitive terms nor to having the character of hints. There is at least one complex function term that Frege both defines and offers 'a few elucidations' to help his readers understand the term.⁸³ These elucidations consist of perfectly straightforward natural language arguments—arguments that are indistinguishable from natural language proofs—that are meant to show us what value the defined function has for different arguments. What is characteristic of elucidation comes out in the next section, where Frege writes

[O]ur elucidation could be wrong in other respects without placing the correctness of those proofs in question; for only the definition itself is the foundation for this edifice.⁸⁴

The character of the discussions and arguments in Frege's writings that play elucidatory roles varies dramatically—from apparently paradoxical remarks that (Frege himself says) must either be false or miss his thought to elaborate arguments that might easily be (and sometimes ultimately are) expressed in *Begriffsschrift*. What marks a discussion as elucidatory is neither its form nor its content but, rather, its role in the project. The mark of elucidation is its contribution to the propaedeutic.

IX

Truth

Let us return, finally, to the issue of the truth predicate. Frege makes at least one general

statement about truth that seems to be neither a gloss on the meaning of Begriffsschrift terms nor assimilable to logical laws. This is the statement with which we have been concerned for most of this paper: that a sentence can have truth-value only if each of its constituents has Bedeutung. How is this to be understood? It purports to distinguish between sentences—those that do, and those that do not, have truth-values. Were this statement a part of a theory of truth (or, a law of truth), one would expect it to be applicable to particular languages. And as we saw earlier, given the criteria a term must satisfy if it is to have Bedeutung, one upshot of this law is that virtually no sentence of natural language has truth-value. If our interest is in natural language, the purported distinction between sentences that do, and sentences that do not, have truth-values does not do any work. The same holds for logically perfect language. For a logically perfect language must be constructed, Frege tells us, so that each of its terms has Bedeutung. All sentential expressions of a logically perfect language will have truth-values. Thus, as a statement of a theory about language and truth, it seems either platitudinous or wrong.

The point of this statement, I have been arguing, is that it is part of Frege's articulation of his standards for introducing a systematic science of logic; part of his articulation of a regulative ideal. And we can see from this why, in spite of the fact that there is no need for a truth predicate in *Basic Laws*, Frege would not want to eliminate the truth predicate from natural language. For there is a continuing role for a truth predicate to play in natural language—even on Frege's view. Our everyday concerns and concepts play a continuing role in introducing new questions to be addressed by scientific research. It will always be the case that most science is in its early stages and not yet systematizable. Science in its early stages requires a language with logical defects—a language, for example, in which it is possible to use a predicate that has no

Bedeutung. But on Frege's view the further development of science requires progressively more rigor and precisification. It requires, in particular, fixing the Bedeutung of the predicate in question. Thus to say that a sentence has a truth-value only if each of its constituents has Bedeutung is to state part of a regulative ideal that, Frege thinks, guides scientific research. The statement of, and attention to, this regulative ideal will continue to have importance for as long as our everyday concerns and concepts motivate the formulation and pursuit of new scientific projects.

¹ My thanks to David Bell, Gary Ebbs, Mark Kaplan, Michael Liston, and Thomas Ricketts for comments at various stages. Versions of parts of this paper were read to the conference on Truth in Frege at the University of London, the Society for Analytic Philosophy at Erlangen-Nürnberg and the philosophy departments at Friedrich-Schiller Universität Jena, Indiana University, Leipzig University, Notre Dame University, Sheffield University, and the University of Illinois at Urbana-Champaign. I am indebted to members of these audiences. I am indebted to the American Philosophical Society and the Bogliasco foundation for support and to the Philosophy Programme, School of Advanced Study at the University of London. This paper contains short versions of arguments from "What's in a Numeral? Frege's Answer", (*Mind* April 2007) and "Semantic Descent" (*Mind* April 2005). The arguments are reprinted with permission.

2. *Foundations* section 3, p. 4.

³ *Foundations* section 3, p. 4.

4. *Foundations* section 3, p. 4.

5. *Foundations* section 4, p.5.

6. Although Frege does not explicitly discuss this, it is obvious that, if the proofs based on an unproved primitive law are to establish the truth of their conclusion, the truth of the primitive law must be evident without proof.

7. This is primitive-eligible, but not a basic Begriffsschrift law. Since Frege wants to minimize the number of primitive laws (see, e.g., *Grundgesetze* p. vi), he derives many laws that are primitive-eligible.

8. *Grundgesetze* pp. xvi-xvii.

9. *Foundations* p. 21.

10. *Foundations* p. 20.

11. *Foundations*, p. 21.

¹² *Foundations* pp. 20-21.

13. *Foundations* p. 21.

14. *Foundations* p. iv.

15. *Foundations* p. 81.

16. *Foundations*, section 75, p. 88; section 78, p. 91.

17. *Foundations*, section 19, p. 26.

18. One might suspect, as Patricia Blanchette argues in "Frege's Reduction", *History and Philosophy of Logic*, 15 (1994), 85-103), that Frege requires statements in the systematic science of arithmetic to be logically equivalent to claims of ordinary pre-systematic arithmetic. I argue below that this interpretation conflicts with Frege's statements about the roles played by Begriffsschrift and natural language.

19. *Foundations*, section 47, p. 60.

20. *Foundations*, section 63, p. 74.

21. *Foundations*, section 74, p. 87.

22. *Foundations*, section 100, p. 110.

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23. *Foundations*, section 103, p. 112
30. *Foundations*, section 109, p. 119.
25. *Foundations*, section 69, p. 80.
26. *Foundations*, section 107, p. 117. Later, Frege attached more importance to bringing in extensions. But his reason is that "we just cannot get on without them" (*Grundgesetze* p. x), not that numbers really are extensions.
- ²⁷ I concentrate here on reference. For a discussion of sense and of Frege's discussion of analytic vs. constructive definitions see my 2007.
28. As Gary Kemp argues in "Frege's Sharpness Requirement", in *Philosophical Quarterly* 46 (1996), 168-184. Many of the following arguments are responses to his objections.
29. *Grundgesetze*, volume ii, § 56.
30. *Grundgesetze*, volume ii, §62.
- ³¹ See the arguments in *Grundgesetze* II sections 56-67.
32. See also the discussion of the universal generalization of ' $(x > 2) \supset (x^2 > 2)$ ' in "Peano's Conceptual Notation".
33. "On the Scientific Justification of Begriffsschrift", P. 51/CN p. 86.
34. *Begriffsschrift* p. v.
35. "Logic in Mathematics", NS p. 261/PW p. 242.
36. See e.g., Frege's discussion, in "On the Scientific Justification of Begriffsschrift" pp. 50-51/CN pp.84-85, of Euclid's tacit presuppositions.
- ³⁷ Sept. 1924. See NS pp. 285/PW p. 266.
38. "On Sense and Meaning", p. 31.
39. "On Sense and Meaning", pp. 31-32.
- ⁴⁰ Frege also warns against apparent proper names without Bedeutung. But the imperfection in question is not that language *has* proper names with no Bedeutung, but rather that it is *possible to form* proper names with no Bedeutung. This possibility cannot be prohibited in natural language.
- ⁴¹ See "Logic in Mathematics", NS p. 239/PW p. 221.
42. Body mass index (or Quetelet index) is defined as: $[\text{weight in kg}]/[\text{height in meters}]^2$.
- ⁴³ Kuczumski and Flegal "Criteria for definition of overweight in transition: background and recommendations for the United States" *American Journal of Clinical Nutrition*, Vol. 72, No. 5, p. 1077
- ⁴⁴ See, e.g., the historical remarks in Robert J Kuczumski and Katherine M Flegal. Myriad internet web pages give examples of athletes who count as obese on this definition (favorite examples are Sylvester Stallone and Arnold Schwarzenegger). See, e.g., <http://www.obesityscam.com/myth1.1.htm>.
- ⁴⁵ Central to the supervaluationist approach is the notion of precisification or a sharpening of the bounds of a predicate. Given a particular precisification of, e.g., the term 'bald', each person is either bald or not bald. On the supervaluationist account of a sentence containing a vague predicate, the sentence is true just in case it is true given any admissible precisification. See, e.g., Kit Fine "Vagueness, Truth and Logic," *Synthese*, XXX (1975).
- ⁴⁶ "Thoughts" p. 59.
- ⁴⁷ For a more thorough discussion, see my "Semantic Descent".
- ⁴⁸ That we need to talk of forms of sentences and truth predicates in order to make general claims (e.g., of infinitely many axioms of the form ' $P \rightarrow P$ ', that they are logical truths) is not just Quine's view. See, e.g., Jason Stanley's claim, in "Truth and Metatheory in Frege", *Pacific Philosophical Quarterly*, volume 77, p. 53 that one reason that a truth predicate occurs ineliminably in discussions of the validity of rules of inference is that they are generalizations.
- ⁴⁹ *Word and Object* p. 273, my emphasis
- ⁵⁰ *Pursuit of Truth*, p. 81; see also *Philosophy of Logic* pp. 11-12.
- ⁵¹ *The Logical Basis of Metaphysics*, p. 23.
- ⁵² Or, for those who favor a model-theoretic view, a relation between sentences and interpretations. Since Frege objects to viewing Begriffsschrift expressions as subject to multiple interpretations. (see, e.g., "Foundations of Geometry" with II, p. 384), a contemporary version of Frege's view would be one on which truth is a property. One might think that the view stated here is already far from Frege's since, especially in his later work, Frege characterizes truth as something that applies to thoughts rather than sentences. However, there is a relevant property of sentences—not that of truth, but that of expressing a truth. This issue does not affect the argument that follows.
- ⁵³ *Philosophy of Logic* p. 11
- ⁵⁴ I focus solely on the second (*Basic Laws*) version of Frege's logic. No argument is needed about the first

(*Begriffsschrift*) version, since there is no candidate for a truth predicate there. Frege does not use the term ‘wahr’ but, rather, ‘bejaht’ (affirms) and ‘verneint’ (denies) and, on occasion, a variety of other terms (e.g., ‘stattfindet’). ‘Bejahen’ is not a truth predicate. It is not applied to sentential expressions but, rather, to sentential expressions prefixed by the judgment stroke.

55. *Grundgesetze*, volume I, p. 25. For convenience I use the arrow rather than Frege’s actual symbols: the horizontal combined with the condition stroke.

⁵⁶ David Bell has pointed out to me, in conversation, that “is the True” is indisputably a truth predicate in this sense: it is a predicate whose only topic is truth. However, what is at issue here is whether “is the True” is the sort of predicate used in contemporary semantics or metatheory. I use the expression “truth predicate” to describe a predicate that is meant to hold either of (all and only) true sentences or of (all and only) true thoughts.

⁵⁷ This, of course, is a long story. For an account see chapter 5 of my *Frege Explained*, (Open Court Press) 2004.

⁵⁸ “Function and Concept”, p. 13.

⁵⁹ There is also, I shall argue (see footnote 69), no use of a predicate that holds of true thoughts.

⁶⁰ “On Concept and Object” p. 193.

⁶¹ *Basic Laws*, p. 7.

⁶² *Basic Laws*, p. 9.

⁶³ *Basic Laws*, pp. 9-10.

⁶⁴ Jamie Tappenden argues that this assertion about the meaning of ‘is the True’ is unjustified. For a response to Tappenden, see my “Semantic Descent”

⁶⁵ Thus, e.g., were the following correct:

‘ $1+1=2$ ’ = the True

and

‘ $2 < 5$ ’ = the True

we could infer that

‘ $1+1=2$ ’ = ‘ $2 < 5$ ’

– i.e., that the sentences are the same. Of course the statements set off above are not correct, on Frege’s view.

Rather, on his view,

$(1+1=2)$ = the True

and

$(2<5)$ = the True.

The consequence that $(1+1=2) = (2 < 5)$ is one that Frege embraces. The same argument also shows that ‘is the True’ is not a predicate that holds of true thoughts (if it were, there would be only one true thought). This is not to say that there is never any use in *Basic Laws* of a predicate that is meant to hold of true sentences. Frege does use such a predicate; it is *bedeutet das Wahre*. But this predicate does not appear in his discussions of his rules of inference and logical laws.

⁶⁶ One might suspect that Frege was simply not as careful as he might have been. After all, Frege *could have* used the predicate that is supposed to hold of all true sentences in his discussion of Modus Ponens. Perhaps Frege simply did not notice the difference between ‘is the True’ and ‘means (or denotes) the True’. For an argument that this is not so, see my “Semantic Descent”.

⁶⁷ However, sometimes Frege *does* use capital Greek letters as metalinguistic variables. Whenever he uses these symbols as metalinguistic variables (see, e.g., the *Grundgesetze* introduction of the identity sign), he uses quotation marks and a predicate, *bedeutet das Wahre*, that is meant to hold of linguistic expressions. In contrast, when he does not (e.g., in the passage quoted above), he uses no quotation marks and the predicate, ‘is the True’ is meant to hold of non-linguistic objects.

⁶⁸ The vertical line that begins this expression is Frege’s judgment stroke. Although it would take us too far afield to discuss his use of this expression in detail here, one feature of its use is that expressions like this expression of Basic Law 1 that are preceded by a judgment stroke are universally quantified. The actual quantifiers inserted in the expressions below, simply represent in more familiar form something that is actually in Frege’s notation.

⁶⁹ This is not to say that Frege is offering a strategy for eliminating the truth predicate from natural language. As I will argue shortly, a natural language truth predicate is useful for Frege’s purposes. Nor is there reason to believe that Frege would object to the use of a truth predicate in a systematic science. But such a science would not be logic. In particular, as I argue in “Semantic Descent”, ‘new science’ Frege discusses in “On the Foundations of

Geometry" is not logic. Moreover, although Frege uses the expressions "denotes the True" and "denotes the False" throughout the early sections of *Basic Laws*, with only two exceptions, he completely avoids them when he is talking about the justification of his basic laws or rules.

⁷⁰ *Grundgesetze*, volume I, pp. xiv-xv.

⁷¹ *Grundgesetze*, volume I p. xvi.

⁷² *Grundgesetze*, volume I p. xvi.

73. *Grundgesetze*, volume I, p. xv.

⁷⁴ *Grundgesetze*, volume I, p. xv, see also, PW pp.145-6.

⁷⁵ See, e.g., *Grundgesetze*, volume I p. xv; see also PW pp. 3, 128.

⁷⁶E.g., in his unpublished articles about Boole's logical notation and Begriffsschrift, Frege argues that Begriffsschrift is superior to Boole's notation. I am indebted to Ian Rumfitt for bringing up the issue of Frege's discussions of Boole.

77. "Foundations of Geometry II", p. 301, see also *Grundgesetze*, volume I, p. 4.

78. "Foundations of Geometry II", p. 301.

⁷⁹ For an argument, see my *Frege in Perspective*, pp. 246-259; see also my *Frege* pp. 105-116.

⁸⁰ *Grundgesetze*, vol I, p. xvii.

81. "Compound Thoughts", p. 50.

82. Some of the discussion below is a response to these objections from Jason Stanley's "Truth and Metatheory in Frege", *Pacific Philosophical Quarterly*, volume 77. I do not discuss Stanley's claim that the discussions of section 31 of *Basic Laws* are meant as metatheoretic proofs. This much is clearly right: section 31 contains natural language discussions about Begriffsschrift, that have the character of argument. But if that is all that is meant by 'metatheory', then metatheory includes what Frege calls 'elucidation' (see the remarks below about section 34) and need not satisfy the standards we apply to proofs. I argue in "Section 31 Revisited: Frege's Elucidations", in E. Reck, *From Frege to Wittgenstein: Perspectives on Early Analytic Philosophy*, Oxford University Press, that while these discussions do make sense as elucidations, they do not make sense as metatheoretic proofs.

⁸³ See, *Basic Laws*, section 34.

⁸⁴ *Grundgesetze* vol I, §35.