

# Q520 Notes on Natural Logic

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We have seen examples of what are traditionally called *syllogisms* already:

$$\frac{\begin{array}{l} \textit{All men are mortal.} \\ \textit{Socrates is a man.} \end{array}}{\textit{Socrates is mortal.}}$$

The idea again is that the sentences above the line should *semantically entail* the one below the line. Specifically, in every context<sup>1</sup> in which *All men are mortal* and *Socrates is a man* are true, it must be the case that *Socrates is mortal* is also true.

What we want to do here is to turn this *semantic* entailment into the a rule in a *formal proof system*.

## 1 The fragment with sentences *All X are Y*.

It will be easier to postpone the introduction of names a bit, so to begin we'll only deal with sentences *All X are Y*. More formally, we have *variables*  $X_1, X_2, \dots, X_n, \dots$  ranging over common nouns. The  $X$ 's denote the  $v$  variables. A *sentence* is an expression  $S$  of the form *All X are Y*, where  $X$  and  $Y$  are one of the  $v$  variables. For example, *All  $X_{12}$  are  $X_{7132}$*  is a sentence in this first fragment.

Next, we need a notion of semantics for this fragment. In this discussion, a *model* (or *interpretation*, or *context*), will be a pair  $(U, \llbracket \cdot \rrbracket)$ , where  $U$  is a set, and  $\llbracket X_i \rrbracket \subseteq U$  for all  $i$ . We then use this to define the truth or falsity of sentences. The definition is

$$\llbracket \textit{All } X \textit{ are } Y \rrbracket = \begin{cases} \text{true} & \text{if } \llbracket X \rrbracket \subseteq \llbracket Y \rrbracket \\ \text{false} & \text{otherwise} \end{cases}$$

If  $\Gamma$  is a set of sentences and  $S$  be a sentence, we say that  $\Gamma$  *semantically implies*  $S$  and we write  $\Gamma \models S$  if: for all contexts  $(U, \llbracket \cdot \rrbracket)$ , if  $\llbracket A \rrbracket = \text{true}$  for all  $A \in \Gamma$ , then also  $\llbracket S \rrbracket = \text{true}$ . For example,

$$\{\textit{All } X_1 \textit{ are } X_2, \textit{All } X_2 \textit{ are } X_3\} \models \textit{All } X_1 \textit{ are } X_3.$$

On the other hand,

$$\{\textit{All } X_1 \textit{ are } X_2, \textit{All } X_3 \textit{ are } X_2\} \not\models \textit{All } X_1 \textit{ are } X_3.$$

What one wants to do in a formal system of logic is to give a purely syntactic counterpart  $\vdash$  to  $\models$ . We read  $\vdash$  as “proves” or “derives”. The goal is to give a definition of a relation  $\Gamma \vdash S$  and then to check that it agrees with the earlier notion  $\Gamma \models S$ .

A *proof tree over*  $\Gamma$  for this fragment is a tree with the following properties:

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<sup>1</sup>In these notes, we use “context” and “model” synonymously.

1. The leaves are either labeled with sentences in  $\Gamma$ , or with sentences of the form *All X are X*.
2. The interior leaves have two children (drawn above them); if the label of the parent (on the bottom) is *All X are Y*, then the label of the left child is *All X are Z*, and the label of the right child is *All Z are Y*.

We draw these trees with the root at the bottom and the leaves at the top.

We summarize this schematically by indicating the *inference rules* as follows:

$$\frac{\frac{\textit{All X are Z} \quad \textit{All Z are Y}}{\textit{All X are Y}}}{\textit{All X are X}}$$

If there is a proof tree over  $\Gamma$  whose root is labeled  $S$ , we say  $\Gamma \vdash S$ . This is how our proof system works.

Here is an example: Let  $\Gamma$  be

$$\{\textit{All A are B}, \textit{All Q are A}, \textit{All B are D}, \textit{All C are D}, \textit{All A are Q}\}$$

Let  $S$  be *All Q are D*. Here is a proof tree showing that  $\Gamma \vdash S$ :

$$\frac{\frac{\frac{\textit{All A are B} \quad \textit{All B are B}}{\textit{All A are B}} \quad \textit{All B are D}}{\textit{All A are D}}}{\textit{All Q are D}}$$

Note that all of the leaves belong to  $\Gamma$  except for one that is *All B are B*. Note also that some elements of  $\Gamma$  are not used as leaves. This is permitted according to our definition. The proof tree above shows that  $\Gamma \vdash S$ . Also, there is a smaller proof tree that does this, since the use of *All B are B* is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All A are A*.)

**Lemma 1.1 (Soundness)** *If  $\Gamma \vdash S$ , then  $\Gamma \models S$ .*

**Proof** To prove this formally, you need the notion of *induction on proof trees*. We didn't really cover this, so our proof will be a bit new. Suppose  $\mathcal{T}$  is a tree that shows  $\Gamma \vdash S$ . If  $\mathcal{T}$  is a one-element tree, let  $S$  be the label on the one node. Either  $S$  belongs to  $\Gamma$ , or else  $S$  is of the form *All A are A*. In the first case, every model satisfying every sentence in  $\Gamma$  clearly satisfies  $S$ , as  $S$  belongs to  $\Gamma$ . And in the second case, every model whatsoever satisfies  $S$ .

So we know our result for one-element trees  $\mathcal{T}$ . In a more general setting, let's suppose that we have a tree  $\mathcal{T}$  that has as immediate subtrees  $\mathcal{T}_1$  and  $\mathcal{T}_2$ . What do the labels look like? Well, the label on  $\mathcal{T}$  must be a sentence *All X are Z*, and then for some  $Y$  the label of the left child is *All X are Y*, and the label of the right child is *All Y are Z*. Now  $\mathcal{T}_1$  and  $\mathcal{T}_2$  are smaller than our  $\mathcal{T}$ . So *we assume the result we are trying to establish about  $\mathcal{T}$  for the smaller trees  $\mathcal{T}_1$  and  $\mathcal{T}_2$* . (This is where induction comes in!) This tells us that  $\Gamma \models \textit{All X are Y}$ , and also  $\Gamma \models \textit{All Y are Z}$ . We claim that  $\Gamma \models \textit{All X are Z}$ . Take any content in which all sentences in  $\Gamma$  are true. Then  $\llbracket X \rrbracket \subseteq \llbracket Y \rrbracket$  by our first point above. And  $\llbracket Y \rrbracket \subseteq \llbracket Z \rrbracket$  by second. So  $\llbracket X \rrbracket \subseteq \llbracket Z \rrbracket$  by transitivity of subset. Since the context here is arbitrary, we conclude that  $\Gamma \models \textit{All X are Z}$ .  $\dashv$

**Theorem 1.2 (Completeness)** *In the fragment with All, if  $\Gamma \models S$ , then  $\Gamma \vdash S$ .*

**Proof** Let  $Z_1, \dots, Z_k$  be all the variables that occur in  $\Gamma$  or in  $S$ . Let  $S$  be *All X are Y*.

Define a model by  $U = \{*\}$ ,<sup>2</sup> and

$$\llbracket Z_i \rrbracket = \begin{cases} U & \text{if } \Gamma \vdash \text{All } X \text{ are } Z_i \\ \emptyset & \text{otherwise} \end{cases} \quad (1)$$

We claim that if  $\Gamma$  contains *All V are W*, then  $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$ . For this, we may assume that  $\llbracket V \rrbracket \neq \emptyset$  (otherwise the result is trivial). So  $\llbracket V \rrbracket = U$ . Thus  $\Gamma \vdash \text{All } X \text{ are } V$ . So we have a proof tree as on the left below:

$$\frac{\begin{array}{c} \vdots \\ \text{All } X \text{ are } V \quad \text{All } V \text{ are } W \end{array}}{\text{All } X \text{ are } W}$$

The tree overall has as leaves *All V are W* plus the leaves of the tree above *All X are V*. Overall, we see that all leaves are labelled by sentences in  $\Gamma$ . This tree shows that  $\Gamma \vdash \text{All } X \text{ are } W$ . From this we conclude that  $\llbracket W \rrbracket = U$ . In particular,  $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$ .

Now our claim implies that the context we have defined makes all sentences in  $\Gamma$  true. So it must make the conclusion true. Therefore  $\llbracket X \rrbracket \subseteq \llbracket Y \rrbracket$ . And  $\llbracket X \rrbracket = U$ , since we have a one-point tree for *All X are X*. Hence  $\llbracket Y \rrbracket = U$  as well. But this means that  $\Gamma \vdash \text{All } X \text{ are } Y$ , just as desired.  $\dashv$

**A Stronger Result** Theorem 1.2 proves the completeness of the logical system. But it doesn't give us all the information we would need to have an efficient procedure to decide whether or not  $\Gamma \vdash S$  in this fragment. For that, we need a little more. Define a relation  $\preceq$  on the variables in question by:  $V \preceq W$  if there is a sequence

$$V = V_0, V_1, \dots, V_k = Z$$

such that for  $i = 0, \dots, k-1$ ,  $\Gamma \vdash \text{All } V_i \text{ are } V_{i+1}$ . We allow the sequence to just have  $V = V_0 = V$ , so that we have  $V \preceq V$  for all  $V$ .

**Lemma 1.3** *Let  $\Gamma$  be a set of sentence in our fragment, and define  $\preceq$  by*

$$U \preceq V \quad \text{iff} \quad \Gamma \vdash \text{All } U \text{ are } V \quad (2)$$

*Then*

1. *For all  $U$ ,  $U \preceq U$ .*
2. *If  $U \preceq V$  and  $V \preceq W$ , then  $U \preceq W$ .*

**Theorem 1.4** *Let  $\Gamma$  be any set of sentences in this fragment, let  $\preceq$  be as above. Let  $X$  and  $Y$  be any variables. Then the following are equivalent:*

1.  $\Gamma \vdash \text{All } X \text{ are } Y$ .

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<sup>2</sup>This just means that  $U$  is some one-element set. It doesn't matter which one-element set. Actually, it doesn't even matter that  $U$  has just one element: *any* non-empty set  $U$  would work.

2.  $\Gamma \models \text{All } X \text{ are } Y$ .
3.  $X \preceq Y$ .

The original definition of the entailment relation  $\Gamma \models S$  involves looking at *all* models of the language. Theorem 1.4 is important because part (3) gives a criterion the entailment relation that is algorithmically sensible. To see whether  $\Gamma \models \text{All } X \text{ are } Y$  or not, we only need to compute  $\preceq$  from  $\Gamma$ . This amounts to constructing  $\preceq$ . This is the reflexive-transitive closure of a relation, so it is computationally very manageable. (In graph theoretic terms, one can make a graph of the variables in question using as the edge relation the relation  $\rightarrow$  given by  $Y \rightarrow Z$  iff  $\Gamma \vdash \text{All } Y \text{ are } Z$ . Then  $\preceq$  just means that there is a path in this graph from  $Y$  to  $Z$ .)

**Exercise 1** Prove Theorem 1.4. [Hint: show (1) $\implies$ (2), (2) $\implies$ (3), and (3) $\implies$ (1). The hard step is (2) $\implies$ (3). For this, assume that  $X \not\preceq Y$ . Build a context which makes all the sentences in  $\Gamma$  true and *All X are Y* false. You'll need to modify (1).]

## 2 Some X are Y

We want to now enrich our language by adding assertions *Some X are Y*. We call these sentences *existentials*, since formalizing them in logic would use the existential quantifier  $\exists$ . We extend our semantics by

$$\llbracket \text{Some } X \text{ are } Y \rrbracket = \begin{cases} \text{true} & \text{if } \llbracket X \rrbracket \cap \llbracket Y \rrbracket \neq \emptyset \\ \text{false} & \text{otherwise} \end{cases}$$

We again write  $\Gamma \models S$  for the semantic entailment relation defined the same way as before.

**Exercise 2** Check that

$$\{\text{Some } X \text{ are } Y, \text{Some } Y \text{ are } Z\} \not\models \text{Some } X \text{ are } Z$$

by building a model in which  $\llbracket X \rrbracket \cap \llbracket Y \rrbracket \neq \emptyset$  and  $\llbracket Y \rrbracket \cap \llbracket Z \rrbracket \neq \emptyset$ , but  $\llbracket X \rrbracket \cap \llbracket Z \rrbracket = \emptyset$ .

We add the following three proof rules to our system:

$$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X} \quad \frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X}$$

$$\frac{\text{All } Y \text{ are } Z \quad \text{Some } X \text{ are } Y}{\text{Some } X \text{ are } Z}$$

At this point we have a new proof system, but we keep the notation  $\Gamma \vdash S$ . If we need to differentiate the proof system here from the previous one, we would need to indicate this somehow in our notation.

**Theorem 2.1 (Completeness)** *In the fragment with All and Some, if  $\Gamma \models S$ , then  $\Gamma \vdash S$ .*

**Proof** Suppose that  $\Gamma \models S$ . There are two cases, depending on whether  $S$  is of the form

*All X are Y* or of the form *Some X are Y*. The cases are handled differently. Since the first is easier, we leave it to you as an Exercise. So we fix  $X$  and  $Y$  as suppose that  $\Gamma \models \textit{Some } X \textit{ are } Y$ . We need a proof of this fact in our system.

List all of the existential sentences in  $\Gamma$  in a list:

$$\textit{Some } V_1 \textit{ are } W_1, \textit{Some } V_2 \textit{ are } W_2, \dots, \textit{Some } V_n \textit{ are } W_n \quad (3)$$

Note that we might have repeats among the  $V$ 's and  $W$ 's, and that some of these might well coincide with the  $X$  and  $Y$  that we are dealing with in this proof. For the universe  $U$  we take  $\{1, \dots, n\}$ , where  $n$  is the number in (3). For each variable  $Z$ , we define

$$\llbracket Z \rrbracket = \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\}. \quad (4)$$

(The relation  $\leq$  is defined in (2).) This defines an model of the language.

Consider a sentence *All P are Q* in  $\Gamma$ . Then  $P \leq Q$ . It follows from (3) and Lemma 1.3 that  $\llbracket P \rrbracket \subseteq \llbracket Q \rrbracket$ . Second, take an existential sentence *Some  $V_i$  are  $W_i$*  on our list in (3) above. Then  $i$  itself belongs to  $\llbracket V_i \rrbracket \cap \llbracket W_i \rrbracket$ , so this intersection is not empty.

These facts imply that all sentences in  $\Gamma$  are made true in our model. Recall that our sentence  $S$  is *Some X are Y*. By our assumption that  $\Gamma \models \textit{Some } X \textit{ are } Y$ ,  $\llbracket X \rrbracket \cap \llbracket Y \rrbracket \neq \emptyset$ . Let  $i$  belong to this set. We have four cases, depending on whether  $V_i \leq X$  or  $V_i \leq Y$ , and whether  $W_i \leq X$  or  $W_i \leq Y$ . One case is when  $V_i \leq X$  and  $W_i \leq Y$ . Recalling that *Some  $V_i$  are  $W_i$*  belongs to  $\Gamma$ , we have a proof tree as follows:

$$\frac{\frac{\frac{\textit{All } V_i \textit{ are } X}{\textit{Some } W_i \textit{ are } X} \quad \frac{\textit{Some } V_i \textit{ are } W_i}{\textit{Some } W_i \textit{ are } V_i}}{\textit{Some } X \textit{ are } W_i}}{\textit{Some } X \textit{ are } Y} \quad \textit{All } W_i \textit{ are } Y}{\textit{Some } X \textit{ are } Y}$$

The other cases are similar. +

**Exercise 3** Complete the proof of Theorem 2.1 by showing that if  $\Gamma$  is a set of sentences in the fragment of this section and  $\Gamma \models \textit{All } X \textit{ are } Y$ , then also  $\Gamma \vdash \textit{All } X \textit{ are } Y$ . You will need to modify the proof of Theorem 1.2 a little bit, since  $\Gamma$  now may have existential sentences.

**Exercise 4** Adapt the proof of Theorem 2.1 to show that if

$$\Gamma \not\vdash \textit{Some } X \textit{ are } Y.$$

then there is a model making all sentences in  $\Gamma$  true and also making *Some X are Y* false. By directly following the construction, the size of the model will be the number of existential sentences in  $\Gamma$ . Can we do better?

1. Show by modifying (4) that we can shrink our model down to one of size at most 2.
2. Show that 2 is the smallest we can get by showing that if we only look at one-element models

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \models \textit{Some } X \textit{ are } Z$$

**Exercise 5** Give an algorithm which takes finite sets  $\Gamma$  in the fragment of this section and also single sentences  $S$  and tells whether  $\Gamma \models S$  or not. [You may be sketchy, as we were in our discussion of this matter at the end of Section 1.]

**Exercise 6** Suppose that one wants to say that *All X are Y* is true when  $\llbracket X \rrbracket \subseteq \llbracket Y \rrbracket$  and also  $\llbracket X \rrbracket \neq \emptyset$ . Then the following rule becomes sound:

$$\frac{\text{All } X \text{ are } Y}{\text{Some } X \text{ are } Y}$$

Show that if we add this rule to our proof system, then we get a complete system for the modified semantics. [Hint: Given  $\Gamma$ , let  $\bar{\Gamma}$  be  $\Gamma$  with all sentences *Some X are Y* such that *All X are Y* belongs to  $\Gamma$ . Show that  $\Gamma \vdash S$  in the modified system iff  $\bar{\Gamma} \vdash S$  in the old system.]

**Exercise 7** What would you do to the system to add sentences of the form *Some X exists*?

### 3 Adding names

We continue by adding names so that we can deal with sentences like *John is a secretary*. To our formal language we add *individual variables*  $X_1, \dots, X_n, \dots$ ; we abbreviate these  $J, M, J_1, \dots, J_n, \dots$ , etc. The sentences we add to the fragment are *J is an X* and *J is M*, where  $J$  and  $M$  are individual variables and  $X$  is a common noun variable. We assume that in a model  $U$ ,  $\llbracket J \rrbracket \in U$ . We have to say when sentences with names are true and when they are false. The natural definition is:

$$\begin{aligned} \llbracket J \text{ is an } X \rrbracket &= \begin{cases} \text{true} & \text{if } \llbracket J \rrbracket \in \llbracket X \rrbracket \\ \text{false} & \text{otherwise} \end{cases} \\ \llbracket J \text{ is } M \rrbracket &= \begin{cases} \text{true} & \text{if } \llbracket J \rrbracket = \llbracket M \rrbracket \\ \text{false} & \text{otherwise} \end{cases} \end{aligned}$$

To get a proof system, we add the remaining rules in Figure 1.

We intend to show the completeness of the logic in Figure 1. For this, we need a definition. Fix a set  $\Gamma$  of sentences in this fragment. Let  $\equiv$  be the relation on names defined by

$$J \equiv M \quad \text{iff} \quad \Gamma \vdash J \text{ is } M. \tag{5}$$

**Lemma 3.1**  $\equiv$  is an equivalence relation.

**Theorem 3.2 (Soundness and Completeness)**  $\Gamma \vdash S$  iff  $\Gamma \models S$ .

**Proof** [Sketch] The soundness half of this result is routine. We omit some of the details which are similar to the completeness proof we have already seen. Assume that  $\Gamma \models S$ . We must show that  $\Gamma \vdash S$ . Again, we have cases as to the syntactic form of  $S$ . Perhaps the most interesting is when  $S$  is *Some X are Y*.

As before, we define  $\leq$  to be from (2). We also have the equivalence relation  $\equiv$  from (5). Let the existential sentences in  $\Gamma$  be listed as in (3). Let the set of equivalence classes of  $\equiv$  be  $[J_1], \dots, [J_m]$ .

$\frac{\frac{\text{All } X \text{ are } Z \quad \text{All } Z \text{ are } Y}{\text{All } X \text{ are } Y}}{\text{All } X \text{ are } X}$	$\frac{}{\text{All } X \text{ are } X}$
$\frac{\text{Some } X \text{ are } Y}{\text{Some } Y \text{ are } X}$	$\frac{\text{Some } X \text{ are } Y}{\text{Some } X \text{ are } X}$
$\frac{\text{All } Y \text{ are } Z \quad \text{Some } X \text{ are } Y}{\text{Some } X \text{ are } Z}$	
$\frac{\text{All } X \text{ are } Y \quad J \text{ is an } X}{J \text{ is a } Y}$	$\frac{J \text{ is an } X \quad J \text{ is a } Y}{\text{Some } X \text{ are } Y}$
$\frac{}{J \text{ is } J}$	$\frac{J \text{ is } M \quad M \text{ is } F}{J \text{ is } F}$
$\frac{M \text{ is } J}{J \text{ is } M}$	$\frac{M \text{ is an } X \quad J \text{ is } M}{J \text{ is an } X}$

Figure 1: The rules of our formal logic for the sentences in this section.

We take  $U$  to be  $\{1, \dots, n\} \cup \{[J_1], \dots, [J_m]\}$ . We assume these sets are disjoint. We define

$$\begin{aligned} \llbracket Z \rrbracket &= \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\} \\ &\cup \{[J] : \text{for some } M \in [J], \Gamma \vdash M \text{ is a } Z\} \end{aligned} \quad (6)$$

To finish defining our context, we take  $\llbracket J \rrbracket = [J]$ . That is, the semantics of  $J$  is the equivalence class  $[J]$ .

It is easy to see that the semantics is monotone in the sense that if  $V \leq W$ , then  $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$ . This implies that all of the universal assertions of  $\Gamma$  are true in our model. The existential assertions in  $\Gamma$  are *Some*  $V_i$  *is*  $W_i$  for  $i \leq n$ , and for each  $i$ , the number  $i$  belongs to  $\llbracket V_i \rrbracket \cap \llbracket W_i \rrbracket$ . Finally, consider a sentence  $J$  *is a*  $Z$  in  $\Gamma$ . Then  $\Gamma \vdash J$  *is a*  $Z$ . So  $\llbracket J \rrbracket = [J] \in \llbracket Z \rrbracket$ . This means that our sentence  $J$  *is a*  $Z$  is true in our context.

We conclude that *Some*  $X$  *are*  $Y$  also is true in this context. If there is some number  $i$  in  $\llbracket X \rrbracket \cap \llbracket Y \rrbracket$ , then the proof of Theorem 2.1 shows that  $\Gamma \vdash$  *Some*  $X$  *are*  $Y$ . The only alternative is when for some  $J$ ,  $[J] \in \llbracket X \rrbracket \cap \llbracket Y \rrbracket$ . By the definition in (6), there are  $M \in [J]$  and  $N \in [J]$  such that  $\Gamma \vdash M$  *is an*  $X$  and  $\Gamma \vdash N$  *is a*  $Y$ . Since  $M \in [J]$  and  $N \in [J]$  we also have  $\Gamma \vdash M$  *is*  $N$ . We exhibit a proof tree over  $\Gamma$ :

$$\frac{\frac{\vdots}{M \text{ is an } X} \quad \frac{\frac{\vdots}{N \text{ is a } Y} \quad \frac{\vdots}{M \text{ is } N}}{M \text{ is a } Y}}{\text{Some } X \text{ are } Y}$$

The vertical dots  $\vdots$  mean that there is some tree over  $\Gamma$  establishing the sentence at the bottom of the dots. So  $\Gamma \vdash$  *Some*  $X$  *are*  $Y$ , as desired.  $\dashv$

**Exercise 8** Read the statement of the Completeness Theorem above. The proof that I gave only covers the cases when  $S$  is of the form *Some*  $X$  *are*  $Y$ . The other cases are easier. Give the arguments in those cases.

## 4 No

In this section, we consider the fragment with *No X are Y* on top of *All X are Y*. In addition to the rules of Section 1, we take the following:

$$\frac{\textit{All X are Z} \quad \textit{No Z are Y}}{\textit{No X are Y}} \quad \frac{\textit{No X are Y}}{\textit{No Y are X}} \quad \frac{\textit{No X are X}}{\textit{No X are Y}} \quad \frac{\textit{No X are X}}{\textit{All X are Y}}$$

The soundness of this system is routine.

**Theorem 4.1 (Completeness)** *In the fragment with All and No, if  $\Gamma \models S$ , then  $\Gamma \vdash S$ .*

**Proof** Fix a set  $\Gamma$ . We construct a model  $U_\Gamma = (U, \llbracket \cdot \rrbracket)$  and then show that  $S$  is true in  $U_\Gamma$  iff  $\Gamma \vdash S$ . We take for  $U$  the set of nonempty sets  $s$  of variables satisfying the following conditions:

1. If  $V \in s$  and  $V \leq W$ , then  $W \in s$ .
2. If  $V, W \in s$ , then  $\Gamma \not\vdash \textit{No V are W}$ .

(Note as a special case of the last condition that if  $V \in s$ , then  $\Gamma \not\vdash \textit{No V are V}$ .) We set

$$\llbracket V \rrbracket = \{s \in U : V \in s\}. \quad (7)$$

We claim that each sentence in  $\Gamma$  is true in  $U_\Gamma$ . Condition (1) implies that if *All V are W* belongs to  $\Gamma$ , then  $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$ . Suppose that *No V are W* belongs to  $\Gamma$ . Let  $s \in \llbracket V \rrbracket$ . Then  $V \in s$ . By condition (2),  $W \notin s$ . So  $s \notin \llbracket W \rrbracket$ . This argument shows that  $\llbracket V \rrbracket \cap \llbracket W \rrbracket = \emptyset$ .

We show that if  $S$  is true in  $U_\Gamma$ , then  $\Gamma \vdash S$ . We first deal with the case that  $S$  is the of the form *All X are Y*. Let

$$s = \{Z : X \leq Z\}.$$

Case I:  $s \notin U$ . Then there are  $V, W \in s$  such that  $\Gamma \vdash \textit{No V are W}$ . In this case,

$$\frac{\frac{\frac{\textit{All X are V} \quad \textit{No V are W}}{\textit{No X are W}}}{\textit{All X are W}} \quad \frac{\textit{No X are X}}{\textit{All X are Y}}}{\textit{All X are Y}} \quad (8)$$

Case II:  $s \in U$ . Then since  $s \in \llbracket X \rrbracket$ , we have  $s \in \llbracket Y \rrbracket$ . (7) tells us that  $Y \in s$ , and so  $\Gamma \vdash \textit{All X are Y}$ , as desired.

This concludes our work when  $S$  is *All X are Y*. Suppose that  $S$  is *No X are Y*. Let

$$s = \{Z : X \leq Z \text{ or } Y \leq Z\}.$$

Note that  $X, Y \in s$ . Then  $s \notin U$ , lest  $s \in \llbracket X \rrbracket \cap \llbracket Y \rrbracket$ . So there are  $V, W \in s$  such that  $\Gamma \vdash \textit{No V are W}$ .

Case I:  $\Gamma \vdash \textit{All X are V}$ , and  $\Gamma \vdash \textit{All Y are W}$ . We have the tree:

$$\frac{\frac{\frac{\textit{All X are V} \quad \textit{No V are W}}{\textit{No X are W}}}{\textit{All Y are W}} \quad \frac{\textit{No X are X}}{\textit{All X are Y}}}{\textit{No X are Y}} \quad (9)$$

Case II:  $\Gamma \vdash \textit{All X are V}$ , and  $\Gamma \vdash \textit{All X are W}$ . In this case, take the proof tree in (8) and change the root to *No X are Y*. +