Q520: Homework on Hopfield Networks   Due: Tuesday, January 15

It's fine to work with others on homework, provided that you understand the solution you turn in. It's also fine to quote results from class.
Please note that there are problems on the back.

1. For each of the following, answer true or false with an explanation:

   a. If $\vec{o}$ and $\vec{o'}$ are neighboring observation patterns in a given Hopfield net, then $\Phi(\vec{o})$ and $\Phi(\vec{o'})$ cannot be equal.

   b. The synchronous algorithm for Hopfield nets never goes into an infinite loop.

   c. The sequential algorithm for Hopfield nets never goes into an infinite loop.

   d. If we run the sequential algorithm starting with one of the stored patterns, we always get the very same pattern back.

[In general, if you think a statement in mathematics is false, you should try to give a counterexample. If the statement is true, you should try to find a proof.]

2. Let's consider a Hopfield net for the two patterns $x^1 = (1, 1, 1, 1)$ and $x^2 = (-1, -1, -1, -1)$.

   a. Find the weights of the net.

   b. Show that the stored patterns x^1 and x^2 are local minima of the energy function $\Phi(\vec{o})$.

   c. Conversely, show that if $\vec{o}$ is a local mininum of $\Phi(\vec{o})$, then $\vec{o}$ must be either x^1 or x^2. So with this x^1 and x^2, the net always converges to one of the patterns that we have stored in it.

   d. Suppose we run the sequential algorithm starting on $(1, 1, -1, -1)$. Let's say that we pick our update vertices randomly. Show that the output pattern can be either x^1 or x^2, and that the choice depends entirely on which update vertex we pick first.

3. Let's say that a list of input vectors x^1, x^2, …, x^n is *storable* if the set of local minima of the Hopfield net for these vectors is *exactly* the vectors on this list.
   Is the list $(1, 1, 1, 1), (-1, -1, -1, -1), (1, 1, -1, -1), (-1, -1, 1, 1)$ storable or not?

Problems 2d and 3 are probably done best by brute force calculation.

4. As you may have noticed already, the energy functions that we have seen all have an interesting property: if $\vec{o} = -\vec{o'}$, then $\Phi(\vec{o}) = \Phi(\vec{o'})$.

a. Why is this the case? Your answer should be a short line of algebra.

b. Use the general fact that $\Phi(\vec{o}) = \Phi(\vec{o'})$ to argue another interesting fact: if $\vec{o}$ is a local minimum of energy, so is $\vec{o'}$. [This part takes a little more thinking than the first part.]

5. Our treatment of Hopfield nets is a little easier than what is usually done. Usually, one has not only weights $w_{i,j}$ but *bias values* θ_i on the nodes of the network. (This is the Greek letter 'theta'.) When it comes time to update the net, we take a node, say node i, and we calculate

$$\theta_i + \sum_j o_j w_{i,j}$$

(So the change what we have already seen is the presence of θ_i.) If this whole sum is positive, we set o_i to be 1; if this whole sum is negative, we set o_i to be -1; it it is 0, we keep o_i just as it is. (So this part of the algorithm does not change.)

In dealing with this kind of net, the appropriate definition of energy is:

$$\Phi(\vec{o}) \quad = \quad -\sum_i \theta_i o_i - \frac{1}{2}\sum_{i,j} w_{i,j} o_i o_j.$$

a. Let $\vec{o}$ and $\vec{o'}$ be neighbors, agreeing at all entries except the ith Then

$$\Phi(\vec{o'}) - \Phi(\vec{o}) \quad = \quad 2o_i(\theta_i + \sum_j o_j w_{i,j}).$$

[Your reasoning here should be quite parallel to what we did in class.]

b. Use part (a) to prove the main fact about these kinds of Hopfield nets:

Lemma Let $\vec{o}$ be the vector at some stage of the algorithm.

If a small step of the algorithm changes $\vec{o}$ to $\vec{o'}$, then we must have $\Phi(\vec{o'}) < \Phi(\vec{o})$.

And if some neighbor $\vec{o'}$ of $\vec{o}$ has smaller energy than $\vec{o}$, then some small step of the algorithm will change $\vec{o}$.

[Again, what you need to do in this case is to re-write the proof from class or the notes, but being sure to use the *new* definition of how the net works, and also the *new* definition of energy.]

6. As we know, the two vectors $x^1 = (1, 1, 1, 1)$ and $x^2 = (1, 1, 1, -1)$ are not exactly storable on a Hopfield net of the kind presented in class. (The problem is that any setting of weights which makes these vectors into local minima of energy also makes their negatives into local minima.) Find a Hopfield net *of the new type* which has x^1 and x^2 as its only set of storable vectors. [Hint: you get to set the θ's.] The point of this problem is that adding the bias values θ allows us to store more sets than we otherwise could.]