Linear Algebra

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Vectors

- Vector Addition and Subtraction

$$c = a + b \Leftrightarrow c_i = a_i + b_i, i = 1,\ldots,n$$
$$c = a - b \Leftrightarrow c_i = a_i - b_i, i = 1,\ldots,n$$

- Multiplication by a Scalar

$$b = \varepsilon a \Leftrightarrow b_i = \varepsilon a_i \ , i = 1,\ldots,n$$

- Vector Transpose

Linear Combinations

$$\alpha \begin{bmatrix} u_1 \\ u_2 \\ \ldots \\ u_m \end{bmatrix} + \beta \begin{bmatrix} v_1 \\ v_2 \\ \ldots \\ v_m \end{bmatrix} = \begin{bmatrix} \alpha u_1 + \beta v_1 \\ \alpha u_2 + \beta v_2 \\ \ldots\ldots \\ \alpha u_m + \beta v_m \end{bmatrix}$$

Vector Inner Product

$$\sigma = x \cdot y = \sum_{i=1}^{n} x_i y_i$$
$$\sigma = u^T v = \sum_{i=1}^{n} u_i v_i$$
$$u^T v = v^T u$$

The L_2 Norm

$$\|x\|_2 = (x_1^2 + x_2^2 + \ldots + x_n^2)^{1/2} = (\sum_{i=1}^{n} x_i^2)^{1/2}$$

$$\|x\|_2 = \sqrt{x \cdot x} = \sqrt{x^T x}$$

Matrix

- Addition and subtraction
- Multiplication by a Scalar
- Matrix Transpose

Matrix: column vectors

$$A = \begin{bmatrix} a_{(1)} \,\Big|\, a_{(2)} \,\Big|\, \ldots \,\Big|\, a_{(n)} \end{bmatrix}$$

$$b = Ax; b_i = \sum_{j=1}^{n} a_{ij} x_j$$

$$\begin{bmatrix} a_{(1)} \,\Big|\, a_{(2)} \,\Big|\, \ldots \,\Big|\, a_{(n)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b \end{bmatrix}$$

$$[m \times n] \quad [n \times 1] = [m \times 1]$$

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & & \ldots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{bmatrix}$$

Linear Independence

- Two vectors are not independent if they lie along the same line.

$$x = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; y = 2x = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Matrix Multiplication

- Multiplying a matrix by a vector is a special case of matrix multiplication where

  y = Ax
- This can be written as:

$$y_i = \sum_{k=1}^{N} a_{kj} x_{j,} i = 1,...,M$$

Alternatively we can see the transformation as linear combination of the columns of A

$$y_i = a_1 x_1 + a_2 x_2 + ... + a_N x_N$$

Coordinate Systems

- The vectors a_i have a special interpretation as a coordinate system or basis for a multidimensional space. For example, in the traditional basis in three dimensions,

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, a_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- This basis is orthogonal, since

$$a_i \cdot a_j = 0$$

for all i and j such that $i \neq j$

- The basis vector allow y to be written as

$$y = a_1 y_1 + a_2 y_2 + a_3 y_3$$

Other Bases

- A non-orthogonal basis would also work.

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & -1 \end{bmatrix}$$

would still allow y to be represented (although the coefficients would of course be different). However the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

would not work because there is no way of representing the third component.

Linear Independence

- Consider two linearly independent vectors, u and v, if a third vector, w, cannot be expressed as a linear combination of u and v, then the set {u,v,w} is linearly independent.

Linear Independence

- To represent n-dimensional vectors, the basis must span the space. A general condition for this is that the columns of A must be linearly independent. Formally this means that the only way you could write

$$a_1 x_{(1)} + a_2 x_{(2)} + \ldots + a_n x_{(n)} = 0$$

would be the case that

$$a_1 = a_2 = \ldots = a_n = 0$$

Linear Independence

$$\begin{bmatrix} \Big| & \Big| & & \Big| \\ x_{(1)} & x_{(2)} & \cdots & x_{(n)} \\ \Big| & \Big| & & \Big| \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- The number of linearly independent vectors is called the rank of the matrix.

- When the rank r is less than the dimension N, the vectors are said to span an r-dimensional subspace.

Change of Bases

- Change of basis, or coordinate transform: If we have data that is defined relative to some basis, we are free to re-map that data into a new basis

$$x^* = Ax$$

- Here A defines our new basis. We can always convert back to the original basis via:

$$x = A^{-1} x^*$$

Eigenvectors

- For any matrix W there are special vectors v such that:

$$Wv = \lambda v$$

v is rescaled by a constant λ. The direction of v is not changed.

- The vectors v are known as eigenvectors, and the associated scalars λ are known as eigenvalues.

Example

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Finding Eigenvectors

- The linear equation Ax = 0 only has a solution (non-trivial) if the columns of A are linearly dependent.
- The columns of A are linearly dependent iff the determinant of A is equal to zero, |A|=0.
- Reminder: the determinant of a 2 x 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is given by |A| = ad - bc

Finding Eigenvectors

For a two-dimensional case:

$$\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \lambda\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

Or

$$\begin{bmatrix} 3-\lambda & 1 \\ 2 & 2-\lambda \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

For this equation to have a solution, the columns of the matrix must be linearly dependent, and thus |W|=0. Thus,

$$(3-\lambda)(2-\lambda)-2=0$$

Finding Eigenvectors

- Substituting $\lambda_1 = 4$ into the equation results in

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

- Only one useful equation in two unknowns. Pick V_1=1. Then V_2=1. Thus the eigenvector associated with $\lambda_1 = 4$ is

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$